4.6 Selective withdrawal of thermally stratified fluid

[References]:

Ivey, G. N.
Monosmith, et. al.

We now extend the analysis in the last section and consider the slow and steady flow of a thermally stratified fluid into a two-dimensional line sink.

![Figure 4.6.1: Sketch of velocity profiles across the layer draining into a line sink. (Adapted from Koh, J. Fluid Mechanics, 24, 1966).](image)

Thermal diffusion and convection now comes into play.
4.6.1 Governing equations

We begin with the general law of mass conservation

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{q}) = \frac{\partial \rho}{\partial t} + \vec{g} \cdot \nabla \rho + \rho \nabla \cdot \vec{q} = 0 \]  

(4.6.1)

In environmental problems the range of temperature variation is within a few tens of degrees. The fluid density varies very little and obeys the following equation of state

\[ \rho = \rho_o [1 - \beta (T - T_o)] \]  

(4.6.2)

where \( T \) denotes the temperature and \( \beta \) the coefficient of thermal expansion which is usually very small. Hence

\[ \frac{\vec{g} \cdot \nabla \rho}{\rho \nabla \cdot \vec{g}} = O \left( \frac{\Delta \rho}{\rho} \right) \ll 1 \]

and

\[ \frac{1}{\rho} \frac{\partial \rho}{\partial t} \sim \frac{\Delta \rho}{\rho} \ll 1 \]

It follows that (4.6.1) is well approximated by

\[ \nabla \cdot \vec{g} = 0 \]  

(4.6.3)

which means that water is essentially incompressible. In two dimensions, we have

\[ u_x + w_z = 0 \]  

(4.6.4)

Next, energy conservation requires that

\[ \frac{\partial T}{\partial t} + \vec{q} \cdot \nabla T = D \nabla^2 T \]  

(4.6.5)

Let

\[ T = \bar{T} + T' \]  

(4.6.6)

where \( \bar{T} \) represents the static temperature when there is no motion, and \( T' \) the motion-induced temperature variation. Therefore,

\[ T - T_o = \left( \bar{T}(z) - T_o \right) + T'(x, z, t) \]  

(4.6.7)

and

\[ \frac{\partial T'}{\partial t} + \vec{q} \cdot \nabla \bar{T} + \vec{q} \cdot \nabla T' = D \nabla^2 \bar{T} + D \nabla^2 T' \]  

(4.6.8)

The static temperature must satisfy

\[ \nabla^2 \bar{T} = 0 \]  

(4.6.9)
In a large lake with depth much smaller than the horizontal extent, the static temperature is essentially uniform horizontally. The Laplace equation reduces to

$$D \frac{d^2 T}{dz^2} = 0, \quad \text{implying} \quad \frac{dT}{dz} = \text{constant} \quad (4.6.10)$$

The dynamic part is then governed by

$$\frac{\partial T'}{\partial t} + u \frac{\partial T'}{\partial x} + w \frac{\partial T'}{\partial z} + w \frac{\partial \bar{T}}{\partial z} = D \nabla^2 T' \quad (4.6.11)$$

The exact equations for momentum balance are, in two dimensions,

$$\rho \left( \frac{\partial u}{\partial t} + \vec{q} \cdot \nabla u \right) = -\frac{\partial p}{\partial x} + \mu \nabla^2 u \quad (4.6.12)$$

$$\rho \left( \frac{\partial w}{\partial t} + \vec{q} \cdot \nabla w \right) = -\frac{\partial p}{\partial z} - g \rho_o \beta (T + T' - T_o) + \mu \nabla^2 w \quad (4.6.13)$$

where $p$ denotes the static part, which must satisfy

$$0 = -\frac{\partial p}{\partial z} - g \rho_o \beta (T - T_o) \quad (4.6.14)$$

Taking the difference of the two preceding equations, we find the equation for the dynamic part

$$\rho \left( \frac{\partial w}{\partial t} + \vec{q} \cdot \nabla w \right) = -\frac{\partial p}{\partial z} + g \rho_o \beta T' + \mu \nabla^2 w \quad (4.6.15)$$

### 4.6.2 Approximation for slow and steady flow

For sufficiently slow flows, inertia terms can be ignored. Expecting that vertical motion is suppressed, we further assume that the vertical length scale $\delta$ is much smaller than the horizontal scale $L$, so that $\partial/\partial x \ll \partial/\partial z$. The 2-D momentum equations can then be simplified to

$$0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial z^2} \quad (4.6.16)$$

$$0 = -\frac{\partial p}{\partial z} + g \beta \rho_o T' + \mu \frac{\partial^2 w}{\partial z^2} \quad (4.6.17)$$

Similarly we can linearize (4.6.11) to get

$$w \frac{dT}{dz} = D \frac{\partial^2 T'}{\partial z^2} \quad (4.6.18)$$

Together (4.6.4), (4.6.18), (4.6.16) and (4.6.17) complete the linearized governing equations.
Eliminating $p$ from (4.6.16) and (4.6.17), we get
\begin{equation}
\mu \frac{\partial^2}{\partial z^2} (u_z - w_x) = g \beta \rho \frac{\partial T'}{\partial x} \tag{4.6.19}
\end{equation}
Since
\begin{align*}
\frac{w}{u} &= O \left( \frac{\delta}{L} \right) \ll 1, \\
\frac{w_x}{u_z} &= O \left( \frac{\delta}{L} \right)^2 \ll 1
\end{align*}
we can omit the second term on the left of (4.6.19). In terms of the stream function defined by
\begin{equation}
u = \psi_z, \quad w = -\psi_x \tag{4.6.20}
\end{equation}
(4.6.19) becomes
\begin{equation}
\frac{\partial^4 \psi}{\partial z^4} = \frac{g \beta \rho \frac{\partial T'}{\partial x}}{\mu} \tag{4.6.21}
\end{equation}
Equation (4.6.18) can be written as
\begin{equation}
\psi_x \frac{dT'}{dz} = D \frac{\partial^2 T'}{\partial z^2} \tag{4.6.22}
\end{equation}
We now have just two equations for two unknowns $\psi$ and $T'$. The boundary conditions are
\begin{equation}
T'u, w \downarrow 0, \quad \text{as} \quad z \uparrow \pm \infty \tag{4.6.23}
\end{equation}
or
\begin{equation}
\psi, \psi_x T \downarrow 0, \quad \text{as} \quad z \uparrow \pm \infty. \tag{4.6.24}
\end{equation}
Let the volume rate of withdrawal be prescribed, we must then require the integral condition:
\begin{equation}
\int_{-\infty}^{\infty} u \, dz = -q, \quad \text{implying} \quad \psi(x, z = \infty) - \psi(x, z = -\infty) = q. \tag{4.6.25}
\end{equation}

4.6.3 Normalization
Let
\begin{equation}
\psi = q \psi^*, \quad T = T^* \tag{4.6.26}
\end{equation}
Physically it is natural to choose the characteristic depth of thermal gradient as the global length scale $L$:
\begin{equation}
L = -\left( \frac{\beta \frac{dT'}{dz}}{dz} \right)^{-1} \tag{4.6.27}
\end{equation}
The scales $T_o$ and $\delta$ are yet to be specified.

The dimensionless (4.6.21) reads

$$\frac{q}{\delta^4} \left( \frac{\partial^4 \psi}{\partial z^4} \right)^* = \frac{g \beta \rho_o T_o}{\mu L} \left( \frac{\partial T'}{\partial x} \right)^* ,$$

hence we choose

$$\frac{q}{\delta^4} = \frac{g \beta \rho_o T_o}{\mu L}$$

so that

$$\left( \frac{\partial^4 \psi}{\partial z^4} \right)^* = \left( \frac{\partial T'}{\partial x} \right)^* .$$

Similarly, (4.6.22) becomes

$$\frac{T_o}{\delta^2} \left( \frac{\partial^2 T'}{\partial z^2} \right)^* + \frac{q}{DL} \frac{dT}{dz} \left( \frac{\partial \psi}{\partial x} \right)^* = 0$$

after normalization, suggesting the choice of

$$\frac{T_o}{\delta^2} = \frac{q dT/\partial z}{DL}$$

so that

$$\left( \frac{\partial^2 T'}{\partial z^2} \right)^* + \left( \frac{\partial \psi}{\partial x} \right)^* = 0 .$$

Eqs. (4.6.28) and (4.6.30) can be solved to give the scales

$$\delta = \frac{L^{1/3}}{\alpha}, \quad \text{where} \quad \alpha = \frac{g \beta \rho_o dT}{d\mu D \partial z}$$

and

$$T_o = q \left( \frac{\delta^2 \partial T/\partial z}{DL} \right) .$$

The flux condition is normalized to

$$\psi^*(\infty) - \psi^*(-\infty) = 1 .$$
4.6.4 Similarity solution

Let us try a one-parameter similarity transformation

\[ x = \lambda^a \hat{x}, \quad z = \lambda^b \hat{z}, \quad \psi = \lambda^c \hat{\psi}, \quad T' = \lambda^d \hat{T} \quad (4.6.35) \]

The exponents \(a, b, c\) and \(d\) will be chosen so that the boundary value problem is formally the same as the original one. To achieve invariance of (4.6.34), we set \(c = 0\). In addition we set

\[ \lambda^{-4b} = \lambda^{d-a} \]

for (4.6.29), and

\[ \lambda^{d-2b} = \lambda^{-a} \]

for (4.6.31). Hence, \(d - a = -4b\) and \(a - 2b = -d\) implying

\[ b = -d, \quad a = 3b = -3d. \quad (4.6.36) \]

These relationships among the exponents suggest the following new similarity variables:

\[ \psi = f(\zeta), \quad T = \frac{h(\zeta)}{x^{1/3}} \quad (4.6.37) \]

with

\[ \zeta = \frac{z}{x^{1/3}} \quad (4.6.38) \]

It is easily verified that these variables are invariant under the similarity transformation. Carrying out the differentiations

\[ \psi_z = \frac{f'}{x^{1/3}}, \quad \psi_{zzzz} = \frac{f'''}{x^{3/3}} \]

\[ T_x = h' \frac{1}{x^{1/3}} \left( -\frac{1}{3} \frac{z}{x^{4/3}} \right) + h \left( -\frac{1}{3} \right) \frac{1}{x^{4/3}} \]

\[ = - \left( \frac{1}{3} \zeta h' + \frac{h}{3} \frac{1}{x^{4/3}} \right) \]

we get from (4.6.29)

\[ f''' = \frac{-1}{3} \left( \zeta h' + h \right) \quad (4.6.39) \]

Since

\[ T_z = h' \frac{1}{x^{2/3}}, \quad T_{zz} = h'' \frac{1}{x} \]

\[ \psi_x = f' \frac{z}{x^{1/3}} \left( -\frac{1}{3} \right) = -\frac{1}{3} f' \zeta \frac{1}{x} \]
we get from (4.6.31)
\[
h'' \frac{1}{x} - \frac{f' \zeta}{3} \frac{1}{x} = 0
\]
or
\[
h'' - \frac{\zeta}{3} f' = 0 \tag{4.6.40}
\]
The boundary conditions are transformed to
\[
f(\infty) - f(-\infty) = -1 \tag{4.6.41}
\]
and
\[
f, f', h \downarrow 0 \quad \text{as} \quad \zeta \to \pm \infty \tag{4.6.42}
\]
Mathematically, the similarity transformation has enabled us to reduce the boundary value problem involving partial differential equations to one with ordinary differential equations (4.6.39), (4.6.40), (4.6.41), and (4.6.42). As long as \( x \) and \( z \) lie on the parabola \( z = \text{const} \ x^{1/3} \), \( \psi^* \) and \( T^* x^{1/3} \) are the same. From the transformation, we can also deduce that the boundary of the zone affected by the flow is a parabola,
\[
\delta \sim x^{1/3} \tag{4.6.43}
\]
Along the centerline \( z = \zeta = 0 \), the velocity varies as
\[
U_{\max} \sim \psi_x \sim x^{-1/3} \tag{4.6.44}
\]
and the temperature varies as
\[
T_{\max} \sim x^{1/3} \tag{4.6.45}
\]
The boundary value problem can now be solved by numerical means (such as Runge-Kutta). Numerical results by Koh (1966, Fig. 4) are shown in Figure (4.6.2).

In Koh (1966), stratification in fluid density is associated with the variation of concentration of a diffusive substance instead of temperature. The fluid density is governed by a diffusion equation formally the same as that for temperature here. To use his numerical results, \( f_0, h_0 \) in his plots are replaced by our \( f, -h \) shown here. Extensive discussion on experimental confirmation as well as the three dimensional theory for a point sink can be found in Koh.
Figure 4.6.2: Temperature and velocity profiles across the layer draining into a line sink.
(Adapted from Koh, J. Fluid Mechanics, 24, 1966).