4.7 Dispersion in an oscillatory shear flow

Relevant to the convective diffusion of salt and/or pollutants in a tidal channel, and chemicals in a blood vessel, Let us examine the Taylor dispersion in an oscillating flow in a pipe. Let the velocity profile be given,

\[ u = U_s(r) + \Re \left[ U_w(r) e^{-i \omega t} \right], \quad 0 < r < a. \tag{4.7.1} \]

The transport equation for the concentration of a solvent is recalled

\[ \frac{\partial C}{\partial t} + \frac{\partial (uC)}{\partial x} = D \left( \frac{\partial^2 C}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C}{\partial r} \right) \right) \tag{4.7.2} \]

Assume the pipe to be so small that diffusion affects the whole radius within one period or so, i.e.,

\[ \tau_0 \sim \frac{2\pi}{\omega} \sim \frac{a^2}{D} \tag{4.7.3} \]

We shall be interested in longitudinal diffusion across \( L \) much greater than \( a \). Let \( U_o \) be the scale of \( U \) and

\[ x = Lx', \quad r = ar', \quad u = U_0u', \quad t = \frac{a^2}{D}t', \quad \Omega = \frac{\omega a^2}{D} \tag{4.7.4} \]

Equation (4.7.2) is normalized to

\[ \frac{\partial C'}{\partial t'} + \frac{Ua}{D} \frac{\partial (u'C')}{\partial x'} = \frac{a^2}{L^2} \frac{\partial^2 C'}{\partial x'^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C'}{\partial r} \right) \tag{4.7.5} \]

Let the Péclét number \( Pe = Ua/D = O(a/L) \) be of (4.7.5) becomes

\[ \frac{\partial C'}{\partial t'} + \epsilon Pe \frac{\partial (u'C')}{\partial x'} = \epsilon^2 \frac{\partial^2 C'}{\partial x'^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C'}{\partial r} \right) \tag{4.7.6} \]

with the boundary conditions

\[ \frac{\partial C'}{\partial r'} = 0, \quad r' = 0, 1 \tag{4.7.7} \]

with

\[ u' = U_s' + \Re U_w' e^{-i \Omega t'} \tag{4.7.8} \]

For brevity we drop the primes from now on.
4.7.1 Multiple scale analysis-homogenization

For convenience let us repeat the perturbation arguments of the last section.

There are three time scales: diffusion time across $a$, convection time across $L$, and diffusion time across $L$. Their ratios are:

$$\frac{a^2}{D} : \frac{L}{U_0} : \frac{L^2}{D} = 1 : \frac{1}{\epsilon} : \frac{1}{\epsilon^2}, \quad (4.7.9)$$

the smallest time scale being comparable to the oscillation period. Upon introducing the multiple time coordinates

$$t, t_1 = \epsilon t, t_2 = \epsilon^2 t \quad (4.7.10)$$

and the multiple scale expansions

$$C = C_0 + \epsilon C_1 + \epsilon^2 C_2 + \ldots \quad (4.7.11)$$

where $C_i = C_i(x, r, t, t_1, t_2)$, then the perturbation problems are

$O(\epsilon^0)$:

$$\frac{\partial C_0}{\partial t} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C_0}{\partial r} \right) \quad (4.7.12)$$

with the boundary conditions:

$$\frac{\partial C_0}{\partial r} = 0, \quad r = 0, 1. \quad (4.7.13)$$

$O(\epsilon)$:

$$\frac{\partial C_0}{\partial t_1} + \frac{\partial C_1}{\partial t} + Pe \frac{\partial (u C_0)}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C_1}{\partial r} \right) \quad (4.7.14)$$

with:

$$\frac{\partial C_1}{\partial r} = 0, \quad r = 0, 1. \quad (4.7.15)$$

$O(\epsilon^2)$:

$$\frac{\partial C_0}{\partial t_2} + \frac{\partial C_1}{\partial t_1} + \frac{\partial C_2}{\partial t} + Pe \frac{\partial (u C_1)}{\partial x} = \frac{\partial^2 C_0}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C_2}{\partial r} \right) \quad (4.7.16)$$

with

$$\frac{\partial C_2}{\partial r} = 0, \quad r = 0, 1. \quad (4.7.17)$$

Ignoring the transient that dies out quickly and focusing attention to the long-time evolution, i.e., $t_1 = O(1)$, the solution at $O(\epsilon^0)$ is

$$C_0 = C_0(x, t_1, t_2), \quad (4.7.18)$$

\footnote{Strictly speaking the solution is}

$$C_0 = C_{00}(x, t_1, t_2) + \sum_{0}^{\infty} C_{0n}(x, t_1, t_2)e^{-\left(k_n^r\right)^2J_0(k_n^r r)}$$

where $k_n^r$ is the $n$-th root of $J_0(ka) = 0$. The series terms die out quickly in $t \gg 1$ and $t_1 \ll 1$ , leaving the limit $C_{00}$ which is independent of $t$. (Dr. E. Qian, 1993)
At $O(\epsilon)$, let the known velocity be

$$u = U_s(y) + \Re \left( U_w(y)e^{-i\omega t} \right)$$  \hspace{1cm} (4.7.19)

then

$$\frac{\partial C_0}{\partial t_1} + \frac{\partial C_1}{\partial t} + Pe \left\{ U_s + \Re \left[ U(r)e^{-i\omega t} \right] \right\} \frac{\partial C_0}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C_1}{\partial r} \right)$$  \hspace{1cm} (4.7.20)

Denoting the period average by overbars,

$$\bar{f} = \frac{\Omega}{2\pi} \int_{t}^{t+2\pi/\Omega} f \, dt$$

and taking the period average,

$$\frac{\partial C_0}{\partial t_1} + PeU_s \frac{\partial C_0}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C_1}{\partial r} \right)$$  \hspace{1cm} (4.7.21)

with

$$\frac{\partial \bar{C}_1}{\partial r} = 0, \quad r = 0, 1$$  \hspace{1cm} (4.7.22)

Let us now integrate (or average) across the pipe, and get

$$\frac{\partial C_0}{\partial t_1} + Pe\langle U_s \rangle \frac{\partial C_0}{\partial x} = 0$$  \hspace{1cm} (4.7.23)

where angle brackets denote averaging over the cross section.

$$\langle h \rangle = \frac{1}{\pi} \int_{0}^{1} 2\pi rh \, dr$$

Now subtract (4.7.23) from (4.7.20)

$$\frac{\partial C_1}{\partial t} + Pe \left\{ \bar{U}_s + \Re \left[ U_w e^{i\omega t} \right] \right\} \frac{\partial C_0}{\partial x} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C_1}{\partial r} \right)$$  \hspace{1cm} (4.7.24)

where

$$\bar{U} = U_s(y) - \langle U_s \rangle$$  \hspace{1cm} (4.7.25)

is the velocity nonuniformity.

Now $C_1$ is governed by a linear equation, we can assume the solution to be proportional to the forcing and composed of a steady part and a time harmonic part, i.e.,

$$C_1 = Pe \frac{\partial C_0}{\partial x} \left\{ B_s(r) + \Re \left[ B_w(r)e^{-i\omega t} \right] \right\}$$  \hspace{1cm} (4.7.26)

then

$$\frac{1}{r} \frac{dB_1}{dr} \left( \frac{dB_1}{dr} \right) = \bar{U}(r)$$  \hspace{1cm} (4.7.27)
and
\[ \frac{1}{r} \frac{d}{dr} \left( r \frac{dB_w}{dr} \right) + i \Omega B_w = U_w(r) \] (4.7.28)
with the boundary conditions
\[ \frac{dB_s}{dr} = 0 \quad \text{and} \quad \frac{dB_w}{dr} = 0, \quad r = 0, 1. \] (4.7.29)

After solving for \( B_s, B_w \) we go to \( O(\epsilon^2) \), i.e., (4.7.16):
\[ \frac{\partial C_0}{\partial t_2} + \frac{\partial C_1}{\partial t_1} + \frac{\partial C_2}{\partial t} \]
\[ + \quad Pe^2 \left\{ \langle U_s \rangle + \dot{U}_s + \Re \left[ U_w e^{-i\Omega t} \right] \right\} \left\{ B_s + \Re \left[ B_w(r)e^{-i\Omega t} \right] \right\} \frac{\partial^2 C_0}{\partial x^2} \]
\[ = \quad \frac{\partial^2 C_0}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C_2}{\partial r} \right) \] (4.7.30)
which is a linear PDE for \( C_2 \). From (4.7.26) and (4.7.23) we find
\[ \frac{\partial C_1}{\partial t_1} = -Pe^2 \frac{\partial^2 C_0}{\partial x^2} \langle U_s \rangle \left\{ B_s(r) + \Re \left[ B_w(r)e^{-i\Omega t} \right] \right\} \] (4.7.31)

It follows that
\[ \frac{\partial C_0}{\partial t_2} + \frac{\partial C_2}{\partial t} \]
\[ + \quad Pe^2 \left\{ \dot{U}_s + \Re \left[ U_w e^{-i\Omega t} \right] \right\} \left\{ B_s + \Re \left[ B_w(r)e^{-i\Omega t} \right] \right\} \frac{\partial^2 C_0}{\partial x^2} \]
\[ = \quad \frac{\partial^2 C_0}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C_2}{\partial r} \right) \] (4.7.32)

Taking the time average over a period,
\[ \frac{\partial C_0}{\partial t_2} + Pe^2 \left\{ \langle \dot{U}_s B_s \rangle + \frac{1}{2} \Re \left[ U_w B_w^* \right] \right\} \frac{\partial^2 C_0}{\partial x^2} = \frac{\partial^2 C_0}{\partial x^2} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial C_2}{\partial r} \right) \] (4.7.33)
with
\[ \frac{\partial C_2}{\partial r} = 0 \quad \text{for} \quad r = 0, 1 \] (4.7.34)

Averaging (4.7.33) across the pipe, we get
\[ \frac{\partial C_0}{\partial t_2} = E \frac{\partial^2 C_0}{\partial x^2} \] (4.7.35)
with
\[ E = 1 - Pe^2 \left\{ \langle \dot{U}_s B_s \rangle + \frac{1}{2} \Re \left( U_w B_w^* \right) \right\} \] (4.7.36)
which is the effective diffusion coefficient or the dispersion coefficient. The first part is of molecular origin; the second part is due to fluid shear.

Finally we add (4.7.23) and (4.7.35) to get:

\[
\left( \frac{\partial}{\partial t_1} + \epsilon \frac{\partial}{\partial t_2} \right) C_0 + Pe\langle U_s \rangle \frac{\partial C_0}{\partial x} = \epsilon E \frac{\partial^2 C_0}{\partial x^2} \tag{4.7.37}
\]

This describes the convective diffusion of the area averaged concentration, which is certainly of practical value.

After the perturbation analysis is complete, there is no need to use multiple scales; we may now write

\[
\frac{\partial C_0}{\partial t_1} + Pe\langle U_s \rangle \frac{\partial C_0}{\partial x} = \epsilon E \frac{\partial^2 C_0}{\partial x^2} \tag{4.7.38}
\]

still in dimensionless form. This equation governs the convective diffusion of the cross-sectional average, after the initial transient is smoothed out.

**Homework**: Find the dispersion coefficient \( E \) in the oscillatory flow in a circular pipe and carry out the necessary numerical calculations.

**Homework (mini research)**: Find the dispersion coefficient \( E \) in the oscillatory flow in a blood vessel with elastic wall.