6.3 Saffman-Taylor instability in porous layer- Viscous fingering

In petroleum recovery water is often used to drive oil from the reservoir. An oil reservoir can also be covered by a layer of water from above. Phenomenon of fingering often occurs when oil is extracted from beneath the water layer. Although known to mining engineers, Saffman & Taylor (1958) gave the first theory and performed simulated experiments in a Hele-Shaw cell.

Consider a moving interface in a stationary coordinate system. Let the initial seepage velocity \( V \) be vertical and the interface be a plane, then

\[
y = \frac{V t}{n}
\]  

(6.3.1)

where \( n \) is the porosity. If the interface is disturbed then its position is at

\[
y = \frac{V t}{n} + \eta(x,t)
\]  

(6.3.2)

At any interior point, \( \phi \) is the velocity potential,

\[
\phi = -\frac{k}{\mu} (p + \rho g y)
\]  

(6.3.3)

where \( k \) is the permeability related to conductivity \( K \) by

\[
K = \frac{\rho g k}{\mu}
\]  

(6.3.4)

The pressure is

\[
p = -\frac{\mu}{k} \phi - \rho g y
\]  

(6.3.5)
Thus in fluid 1 (upper fluid)

\[ p_1 = -\frac{\mu_1}{k_1} \rho_1 g y - \rho_1 g y \]  

(6.3.6)

By continuity,

\[ \nabla^2 \phi_1 = 0, \quad y > \frac{V t}{n} + \eta(x, t), \]  

(6.3.7)

In the lower fluid (2),

\[ p_2 = -\frac{\mu_2}{k_2} \rho_2 g y \]  

(6.3.8)

and

\[ \nabla^2 \phi_2 = 0, \quad y < \frac{V t}{n} + \eta(x, t) \]  

(6.3.9)

Let us first examine the basic uniform flow where the interface is plane (\( \eta = 0 \)). The potentials are

\[ \phi_1^o = V y + f_1(t) = -\frac{k_1}{\mu_1} (p_1^o + \rho_1 g y) \]  

(6.3.10)

\[ \phi_2^o = V y + f_2(t) = -\frac{k_2}{\mu_2} (p_2^o + \rho_2 g y) \]  

(6.3.11)

Note that an arbitrary function of \( f(t) \) is added to the potential without affecting the velocity field. The pressures are

\[ p_1^o = -\left( \frac{\mu_1 V}{k_1} + \rho_1 g \right) y - \frac{\mu_1 f_1(t)}{k_1}, \quad y > \frac{V t}{n} \]  

(6.3.12)

and in the lower fluid (2),

\[ p_2^o = -\left( \frac{\mu_2 V}{k_2} + \rho_2 g \right) y - \frac{\mu_2 f_2(t)}{k_2}, \quad y < \frac{V t}{n} \]  

(6.3.13)

In order that pressure is continuous at \( y = V t/n \) for all \( t \), we must have

\[ f_1(t) = F_1 t, \quad f_2(t) = F_2 t \]  

(6.3.14)

where \( F_1, F_2 \) are constants and

\[- \left( \frac{\mu_1 V}{k_1} + \rho_1 g \right) \frac{V}{n} - \frac{\mu_1 F_1}{k_1} = - \left( \frac{\mu_2 V}{k_2} + \rho_2 g \right) \frac{V}{n} + \frac{\mu_2 F_2}{k_2} \]

Thus

\[ \frac{\mu_2 F_2}{k_2} - \frac{\mu_1 F_1}{k_1} = - \frac{V}{n} \left( \frac{\mu_2}{k_2} - \frac{\mu_1}{k_1} \right) V + \rho_2 - \rho_1 g \]  

(6.3.15)

Note that it is only the difference that matters.

We now consider a small disturbance on the interface .

\[ y = \frac{V t}{n} + \eta(x, t) \]  

(6.3.16)
where

$$\eta = ae^{i ox - i\omega t}$$  \hfill (6.3.17)

is small. The total solution is

$$\phi_1 = Vy + F_1 t + B_1 e^{i ox - \alpha(y - Vt/n) - i\omega t}, \quad y > \frac{Vt}{n} + \eta(x, t)$$  \hfill (6.3.18)

$$\phi_2 = Vy + F_2 t + B_2 e^{i ox + \alpha(y - Vt/n) - i\omega t}, \quad y < \frac{Vt}{n} + \eta(x, t)$$  \hfill (6.3.19)

The linearized kinematic boundary condition is that velocities must be continuous.

$$\frac{\partial \eta}{\partial t} = \frac{\partial \phi_1}{\partial y} \big|_{y=Vt/n} = \frac{\partial \phi_2}{\partial y} \big|_{y=Vt/n}$$  \hfill (6.3.20)

Thus

$$-i\omega n ae^{i ox - i\omega t} = -\alpha a B_1 e^{i ox - i\omega t} = \alpha a B_2 e^{i ox - i\omega t}$$  \hfill (6.3.21)

hence,

$$B_1 = -B_2 = \frac{i\omega n a}{\alpha}$$  \hfill (6.3.22)

Now we require continuity of pressure on \(y = Vt/n + \eta\),

$$-\frac{\mu_1}{k_1} \left( V\eta + \frac{i\omega n \eta}{\alpha} \right) - \rho_1 g\eta = -\frac{\mu_2}{k_2} \left( V\eta - \frac{i\omega n \eta}{\alpha} \right) - \rho_2 g\eta$$  \hfill (6.3.23)

Eliminating \(\eta\) we get

$$i\omega = \frac{\alpha (\rho_2 - \rho_1) g + V \left( \frac{\mu_2}{k_2} - \frac{\mu_1}{k_1} \right)}{n \frac{\mu_2}{k_2} + \frac{\mu_1}{k_1}}$$  \hfill (6.3.24)

Clearly \(i\omega\) is real. If \(i\omega > 0\), or

$$\left( \rho_2 - \rho_1 \right) g + V \left( \frac{\mu_2}{k_2} - \frac{\mu_1}{k_1} \right) > 0$$  \hfill (6.3.25)

the flow is stable. If \(i\omega < 0\), or

$$\left( \rho_2 - \rho_1 \right) g + V \left( \frac{\mu_2}{k_2} - \frac{\mu_1}{k_1} \right) < 0,$$  \hfill (6.3.26)

the flow is unstable.

From the simple model of a tubular porous medium,

$$K = \frac{n \rho g R^2}{8\mu} = \frac{\rho g k}{\mu}$$  \hfill (6.3.27)

hence

$$k = \frac{n R^2}{8}$$  \hfill (6.3.28)
is independent of viscosity and depends only on \( n \) and the pore size. Assume therefore \( k_1 = k_2 \) and that oil (lighter more viscous) lies above water \( \rho_1 < \rho_2 \) and \( \mu_1 > \mu_2 \). If \( V < 0 \) (water pushed downward by oil) then the flow is always stable. Consider \( V > 0 \). The flow is unstable if

\[
V > V_c = \frac{(\rho_2 - \rho_1)g}{\left(\frac{\mu_1}{k_1} - \frac{\mu_2}{k_2}\right)}
\]  

(6.3.29)

Too high an extraction rate causes instability which marks the onset of fingers.

If the water layer is on top of the oil layer, then \( \rho_2 - \rho_1 < 0 \); the flow is unstable even if \( V = 0 \). Since \( \mu_2/k_2 - \mu_1/k_1 > 0 \) a downward flow (water toward oil) is always unstable. A upward flow can be unstable if

\[
0 < V < V_c = \frac{(\rho_1 - \rho_2)g}{\left(\frac{\mu_2}{k_2} - \frac{\mu_1}{k_1}\right)}
\]  

(6.3.30)

Note also that the growth (decay) rate is higher for shorter waves.

A gallery of beautiful photographs of fingering taken from Hele-Shaw experiments can be found in the survey by Homsy.