6.5 Geothermal Plume

Consider a steady, two-dimensional plume due to a source of intense heat in a porous medium. From Darcy’s law:
\[
\frac{\mu}{k}u = -\frac{\partial p}{\partial x} \tag{6.5.1}
\]
where \(k\) denotes the permeability, and
\[
\frac{\mu}{k}w = -\frac{\partial p}{\partial z} - \rho g \tag{6.5.2}
\]
These are the momentum equations for slow motion in a porous medium. Mass conservation requires
\[
u_x + w_z = 0 \tag{6.5.3}
\]
Energy conservation requires
\[
u \frac{\partial T}{\partial x} + w \frac{\partial T}{\partial z} = \alpha \left( \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2} \right) \tag{6.5.4}
\]
where
\[
\alpha = \frac{K}{\rho_0 C} \tag{6.5.5}
\]
denotes the thermal diffusivity.

Equation of state:
\[
\rho = \rho_0 (1 - \beta (T - T_0)) \tag{6.5.6}
\]

Consider the flow induced by a strong heat source. Let
\[
T - T_0 = T', \quad p = p_0 + p'
\]
where \(p_0\) is the hydrostatic pressure satisfying
\[
-\frac{\partial p_0}{\partial z} - \rho_0 g = 0.
\]

Eqn. (6.5.2) can be written
\[
\frac{\mu}{k}w = -\frac{\partial p'}{\partial z} + g \rho_0 \beta T'. \tag{6.5.7}
\]
6.5.1 Boundary layer approximation

Eliminating $p'$ from Eqns. (6.5.7) and (6.5.1), we get

$$\frac{\mu}{k} (w_x - u_z) = g\rho_0 \beta T'_x.$$  

Let $\psi$ be the stream function such that

$$u = \psi_z, \quad w = -\psi_x$$

then

$$\psi_{xx} + \psi_{zz} = -\frac{g\rho_0 k}{\mu} T'_x$$  

(6.5.8)

For an intense heat source, we expect the plume to be narrow and tall. Let us apply the boundary layer approximation and check its realm of validity later,

$$u \ll w, \quad \frac{\partial}{\partial x} \gg \frac{\partial}{\partial z}.$$  

hence

$$\psi_{xx} \approx -\frac{\rho_0 \beta k}{\mu} T'_z$$

or

$$\psi_x \approx -\frac{g\rho_0 \beta k}{\mu} T'_z,$$  

(6.5.9)

which is the same as ignoring $\partial p'/\partial z$ in Eqn. (6.5.7).

This can be confirmed since $u \ll w, \partial p'/\partial x \approx 0$, $p'$ inside the plume is the same as that outside the plume. But

$$\frac{\partial p'}{\partial z} = 0$$

outside the plume, hence $\partial p'/\partial z \approx 0$ inside as well.

Applying the B.L. approximation to Eqn. (6.5.4)

$$uT'_x + wT'_z = \alpha T'_{xx}$$  

(6.5.10)

Using the continuity equation we get

$$(uT')_x + (wT')_z = \alpha T'_{xx}.$$  

Integrating across the plume,

$$\frac{\partial}{\partial z} \int_{-\infty}^{\infty} w T' \, dx = 0$$  

(6.5.11)

since $T' = 0$ outside the plume. It follows that

$$\rho_0 C \int_{-\infty}^{\infty} w T' \, dx = -\rho_0 C \int_{-\infty}^{\infty} \psi_x T' \, dx = Q = \text{constant}.  

(6.5.12)$$
6.5.2 Normalization

Let us take
\[ x = B\bar{x}, \quad z = H\bar{z}, \quad u = \frac{WB}{H}\bar{u}, \quad w = \bar{w}, \quad T' \to \Delta T\theta \]  
(6.5.13)

where \( H, B, \Delta T \) and \( W \) are to be determined to get maximum simplicity. We then get from the momentum equation,
\[ \bar{w} = \bar{\psi}_x = -\frac{g\rho_0\beta\Delta T}{\mu W}\theta, \]
from the energy equation,
\[ \bar{u}\theta_x + \bar{w}\theta_z = \frac{\alpha H}{WB^2}\theta_x, \]
and from the total flux condition,
\[ \rho_0CB\Delta \int_{-\infty}^{\infty} \bar{w}\theta d\bar{x} = Q \]

Let us choose
\[ \frac{g\rho_0\beta\Delta T}{\mu W} = 1 \]  
(6.5.14)

\[ \frac{\alpha H}{WB^2} = 1 \]  
(6.5.15)

and
\[ \rho_0CB\Delta T = Q, \]  
(6.5.16)

which gives three relations among four scales, \( B, H, W, \Delta T \). Then
\[ \bar{w} = \bar{\psi}_x = -\theta, \]  
(6.5.17)
from the energy equation,
\[ \bar{u}\theta_x + \bar{w}\theta_z = \theta_x, \]  
(6.5.18)
and from the total flux condition,
\[ \int_{-\infty}^{\infty} \bar{w}\theta d\bar{x} = 1 \]  
(6.5.19)

In addition we require that
\[ w(\pm\infty, z) = 0, \quad \theta(\pm\infty, z) = 0 \]  
(6.5.20)

\[ u(0, z) = \frac{\partial w(0, z)}{\partial x} = 0, \quad x = 0. \]  
(6.5.21)

From here on we omit overhead bars in all dimensionless equations for brevity.
6.5.3 Similarity solution

Now let
\[ x = \lambda^a x^*, \quad z = \lambda^b z^* \quad \psi = \lambda^c \psi^* \quad \theta = \lambda^d \theta^*. \]

From Eqn. (6.5.17)
\[ \lambda^{c-a} \left( \frac{\partial \psi^*}{\partial x^*} \right) = -\lambda^d \theta^*. \]

For invariance we require,
\[ c - a = d. \tag{6.5.22} \]

From (6.5.19)
\[ -\int \frac{\partial \psi^*}{\partial x^*} dx^* \lambda^{c-a+a+d} = 1. \]

therefore,
\[ a + d = 0. \tag{6.5.23} \]

From Eqn. (6.5.18)
\[ \lambda^{c+d-a-b} = \lambda^{d-2a}. \]

implying,
\[ c + a - b = 0. \tag{6.5.24} \]

Finally
\[ c = \frac{a}{2}, \quad d = -\frac{a}{2}, \quad b = \frac{3}{2} a. \]

In view of these we introduce the following similarity variables,
\[ \eta = \frac{x}{z^{2/3}}, \quad \psi = z^{1/3} f(\eta), \quad \theta = z^{-1/3} h(\eta). \tag{6.5.25} \]

Note that at the center line \( \eta = 0 \)
\[ w = -\psi_x \propto z^{1/3} f'(0)(-)z^{-2/3} \sim z^{-1/3} f'(0) \sim z^{-1/3} \]
\[ \theta \propto z^{-1/3} h(0) \]
\[ b \propto z^{2/3} \tag{6.5.28} \]

Thus the velocity and temperature along the centerline decay as \( z^{-1/3} \) and the plume width grows as \( z^{2/3} \).

Substituting these into Eqns. (6.5.17) and (6.5.18), we get, after some algebra
\[ -\frac{df}{d\eta} = h \tag{6.5.29} \]
and
\[ \frac{df}{d\eta}(fh) = 3 \frac{d^2 h}{d\eta^2}. \tag{6.5.30} \]
The boundary conditions are,

\[
\begin{align*}
    f &= 0 \quad (\psi = 0) \\
    f''(0) &= 0, \quad (w(0, z) = w_{\text{max}}) \\
    f(\pm \infty), \ f'(\pm \infty) &= 0 \\
    h(\pm \infty) &= 0.
\end{align*}
\]

Integrating Eqn. (6.5.30), we get

\[
f h = 3 h'.
\]

Using Eqn. (6.5.29), we get

\[
f f' = 3 f''.
\]

Integrating again, we get

\[-6 f' = f_0^2 - f^2\]

where \(f_0 = f_{\text{max}}\). Let \(f = -f_0 F\), then

\[
f_0 (1 - F^2) = 6 F', \quad \text{or} \quad \frac{dF}{1 - F^2} = \frac{f_0 d\eta}{6}
\]

which can be integrated to give

\[
f_0 \eta \frac{6}{2} = \frac{1}{2} \ln \frac{1 + F}{1 - F}
\]

Thus

\[
\left(\frac{1 + F}{1 - F}\right)^{1/2} = e^{f_0 \eta/6}
\]

or

\[
\left(\frac{1 + F}{1 - F}\right) = e^{f_0 \eta/3}
\]

Solving for \(F\), we get

\[
F = \frac{e^{f_0 \eta/3} - 1}{e^{f_0 \eta/3} + 1} = \tanh \frac{f_0 \eta}{6} \quad (6.5.31)
\]

What is \(f_0\)? Let us use Eqn. (6.5.29)

\[-\int_{-\infty}^{\infty} \frac{df}{d\eta} h d\eta = \int_{-\infty}^{\infty} (f')^2 d\eta = 1
\]

since

\[f' = -f_0 F' = -\frac{f_0^2}{6} \text{sech}^2 \frac{f_0 \eta}{6}\]

and

\[h = -f'.\]
Therefore,
\[
\left( \frac{f_0^2}{6} \right)^2 \int_{-\infty}^{\infty} \text{sech}^4 \left( \frac{f_0 \eta}{6} \right) d\eta = \frac{f_0^3}{6} \int_{-\infty}^{\infty} \text{sech}^4 \zeta d\zeta = 1.
\]

Since
\[
\int_{-\infty}^{\infty} \text{sech}^4 z \, dz = 4/3.
\]
we get \( f_0 \! = \! \left( \frac{9}{2} \right)^{1/3} \) (6.5.32)

The solution is
\[
f = \left( \frac{9}{2} \right)^{1/3} \tanh \left( \frac{9}{2} \right)^{1/3} \frac{\eta}{6}
\]
and
\[
h = -f' = -\left( \frac{9}{2} \right)^{2/3} \text{sech}^2 \left( \frac{9}{2} \right)^{1/3} \frac{\eta}{6}
\]
(6.5.34)

Computed results are given in Figures.

Remark Checking the boundary layer approximation.

\[
\frac{\partial^2 \psi}{\partial x^2} \sim z^{-1}, \quad \frac{\partial^2 \psi}{\partial z^2} \sim z^{-5/3}
\]
\[
\frac{\partial^2 T'}{\partial x^2} \sim z^{-5/3}, \quad \frac{\partial^2 T'}{\partial z^2} \sim z^{-7/3}
\]
hence for large \( z \), B. L. approximation is good.

### 6.5.4 Return to physical coordinates

Start from
\[
\eta = \frac{x}{z^{2/3}}
\]
(6.5.35)
\[
\frac{\bar{\psi}}{z^{1/3}} = f(\eta)
\]
(6.5.36)
\[
z^{1/3} h = h(\eta)
\]
(6.5.37)

Then
\[
\eta = \frac{x}{B \left( z/H \right)^{2/3}} = \left( \frac{H^{2/3}}{B} \right) \left( \frac{x}{z^{2/3}} \right)
\]
(6.5.38)

By eliminating \( H \) and \( \Delta T \) from (6.5.35) and (6.5.37), we get
\[
W = \sqrt{\frac{Qg\beta}{CB}}
\]
From (6.5.36), we get

\[
\frac{H}{B^2} = \frac{W}{\alpha} = \frac{1}{\alpha} \sqrt{\frac{Qg\beta}{CB}}
\]

It follows that

\[
\frac{H}{B^{3/2}} = \frac{1}{\alpha} \sqrt{\frac{Qg\beta}{C}}
\]  \hspace{1cm} (6.5.39)

Now

\[
\frac{\bar{\psi}}{z^{1/3}} = \frac{\psi}{WB} \left( \frac{z}{H} \right)^{-1/3} = \left( \frac{H^{1/3}}{WB} \right) \left( \frac{\psi}{z^{1/3}} \right)
\]  \hspace{1cm} (6.5.40)
It can be shown that
\[
\frac{H^{1/3}}{WB} = \frac{1}{\sqrt{\frac{Q g \beta}{C}}} \left(\frac{H}{B^{3/2}}\right)^{1/3} = \frac{1}{\alpha^{1/3}} \left(\frac{C}{Q g \beta}\right)^{1/3}
\]
which depends on the fluid properties and the given heat source strength.

Also
\[
\bar{z}^{1/3} = h(\eta) = (H z)^{1/3} \Delta T \tau = (H^{1/3} \Delta T) z^{1/3} T'
\]
We can show that
\[
H^{1/3} \Delta T = \frac{1}{\nu} \frac{1}{\sqrt{g \beta C}} \left(\frac{1}{\alpha} \sqrt{\frac{Q g \beta}{C}}\right)^{1/3} = \frac{Q^{1/6}}{\nu(\alpha g \beta)^{1/3} C^{2/3}}
\]
which also depends on the fluid properties and the given heat source strength.