7.6 Transient longshore wind

[Ref]: Chapter 14, p. 195 ff, Cushman-Roisin
Csanady: Circulation in the Coastal Ocean

![Figure 7.6.1: Longshore wind](image)

In view of the last section, we ignore the bottom stress. Assume that the wind is uniform in space but transient in time, so that $\partial/\partial y = 0$, The flux equations are

$$\frac{\partial \eta}{\partial t} + \frac{\partial U}{\partial x} = 0 \quad (7.6.1)$$

$$\frac{\partial U}{\partial t} - fV = -gh \frac{\partial \eta}{\partial x} \quad (7.6.2)$$

$$\frac{\partial V}{\partial t} + fU = \frac{\tau_y^S}{\rho} \quad (7.6.3)$$

The boundary condition on the coast $x = 0$ : $U = 0$.

7.6.1 Sudden long-shore wind

Let the wind stress be

$$\tau_y^S = \begin{cases} 0, & t \leq 0, \\ T, & t > 0. \end{cases} \quad (7.6.4)$$

the initial conditions are

$$\eta, U, V = 0, \quad t = 0, \quad \forall x. \quad (7.6.5)$$
This initial-boundary value problem can be solved by Laplace transform (Crépon, 1967). The solution consists of two parts: one part is oscillatory and decays with time; the other part increases monotonically with time. To avoid the complex mathematics we only examine the latter which is the dominant part for large time,

\[ U = U(x), \quad V = t\bar{V}(x), \quad \eta = t\bar{\eta}(x) \quad (7.6.6) \]

The oscillatory part is needed to ensure the initial condition on \( U \).

It is easy to see from (7.6.1) to (7.6.3) that

\[ \bar{\eta} + \frac{d\bar{U}}{dx} = 0 \quad (7.6.7) \]

\[ f\bar{V} = gh\frac{d\bar{\eta}}{dx} \quad (7.6.8) \]

\[ \bar{V} + f\bar{U} = T/\rho \quad (7.6.9) \]

These three equations can be combined into one:

\[ \frac{d^2\bar{U}}{dx^2} - \frac{f^2}{gh}\bar{U} = -\frac{fT}{\rho gh} \quad (7.6.10) \]

The solution satisfies no flux on the coast is

\[ \bar{U} = \frac{T}{\rho f} \left(1 - e^{-x/R_o}\right) \quad (7.6.11) \]

where

\[ R_o = \frac{\sqrt{gh}}{f} \quad (7.6.12) \]

is called the Rossby radius of deformation. Since \( f = 10^{-4} \) 1/s, in a shallow sea of \( h = 10 \) m the Rossby radius is about \( 10^5 \) m = 100 km.

It is easy to find that

\[ \eta = t\bar{\eta} = -t\frac{T}{\rho gh} e^{-x/R_o} \quad (7.6.13) \]

and

\[ V = t\bar{V} = \frac{T}{\rho f} e^{-x/R_o} \quad (7.6.14) \]

Clearly when \( x/R_o \gg 1 \), the coast line has no influence. The flux is \( U = T/\rho f, V = 0 \), and is inclined to the right of the wind by 90 degrees, as predicted by the Ekman layer theory. The sea surface sinks near the coast if \( T > 0 \) (coast is on the left of wind) and rises if \( T < 0 \) (coast is on the right of wind).
7.6.2 Sinusoidal wind stress

We now consider

\[ \tau_y = \Re \left( \tau_0 e^{-i\omega t} \right) = \tau_0 \sin \omega t \]  

(7.6.15)

Let

\[ (\eta, U, V) = \Re \left[ (\eta_0, U_0, V_0) e^{-i\omega t} \right] \]  

(7.6.16)

The symbol \( \Re \) (real part of) will be omitted for brevity.

Let us calculate the total flux (The boundary layers can be studied later.),

\[ -i\omega \eta_0 + \frac{dU_0}{dx} = 0 \]  

(7.6.17)

\[ -i\omega U_0 - fV_0 = -gH \frac{d\eta_0}{dx} \]  

(7.6.18)

\[ -i\omega V_0 + fU_0 = \frac{i\tau_0}{\rho} \]  

(7.6.19)

An equation for a single variable can be obtained. For example by solving Eqns. (7.6.18) and (7.6.19) for \( U_0 \) and \( V_0 \), we get

\[ U_0 = \frac{-gH \frac{d\eta_0}{dx} - f}{i\tau_0 - i\omega} = \frac{\frac{i\omega gh \frac{d\eta_0}{dx} + i\tau_0 f}{-\omega^2 + f^2}}{f^2 - \omega^2} \left( \frac{d\eta_0}{dx} + \frac{\tau_0 f}{\rho \omega gh} \right) \]  

(7.6.20)

Differentiate Eqn. (7.6.20) and use Eqn. (7.6.17)

\[ -i\omega \eta_0 + \frac{i\omega gh \frac{d^2\eta_0}{dx^2}}{f^2 - \omega^2} = 0 \]

or

\[ \frac{d^2\eta_0}{dx^2} - \frac{f^2 - \omega^2}{gh} \eta_0 = 0 \]  

(7.6.21)

We now distinguish two cases.
**Low frequency: \( \omega < f \)**

The solution to (7.6.21) bounded at infinity is

\[
\eta_0 = A e^{-x/R_0} \tag{7.6.22}
\]

where

\[
R_0 = \sqrt{\frac{gh}{f^2 - \omega^2}}. \tag{7.6.23}
\]

is the modified Rossby radius.

Applying the B.C. on the coast: \( U_0 = 0 \), we get from (7.6.20),

\[
\frac{d\eta_0}{dx} = -\frac{f}{\rho g H} \frac{\tau_0}{\omega} - A \frac{R_0}{R_0}.
\]

and,

\[
A = \frac{\tau_0}{\omega} \frac{f}{\rho g H} R_0.
\]

Hence

\[
\eta_0 = \frac{f \tau_0}{\rho \omega g H} R_0 e^{-x/R_0}
\]

and finally

\[
\eta = \frac{f \tau_0}{\rho \omega g H} R_0 e^{-x/R_0} e^{-i\omega t} \tag{7.6.24}
\]

Now

\[
\eta_t = -i \frac{f \tau_0}{\rho \omega g H} R_0 e^{-x/R_0} e^{-i\omega t} = -U_x.
\]

from Eqn. (7.6.1). Integrating with respect to \( x \),

\[
U = i \frac{f \tau_0 R_0^2}{\rho g h} \left( 1 - e^{-x/R_0} \right) e^{-i\omega t}. \tag{7.6.25}
\]

From Eqn. (7.6.20)

\[
-i \omega V_0 = -f U_0 + i \tau_0 / \rho
\]

\[
= -i \frac{f^2 \tau_0 R_0^2}{\rho g H} \left( 1 - e^{-x/R_0} \right) + \frac{i \tau_0}{\rho}
\]

\[
= \frac{i \tau_0}{\rho} \left[ 1 - \frac{f^2}{g h R_0^2} \left( 1 - e^{-x/R_0} \right) \right]
\]

\[
V_0 = -\frac{\tau_0}{\rho \omega} \left[ 1 - \frac{f^2}{f^2 - \omega^2} \left( 1 - e^{-x/R_0} \right) \right]
\]
Let us summarize the results in real form,

\[ \tau_y^S = \tau_0 \sin \omega t \]  \hspace{1cm} (7.6.26)
\[ \eta = \frac{f \tau_0}{\rho \omega g} R_0 e^{-x/R_0} \cos \omega t \]  \hspace{1cm} (7.6.27)
\[ U = \frac{f \tau_0 R_0^2}{\rho g h} \left(1 - e^{-x/R_0}\right) \sin \omega t \]  \hspace{1cm} (7.6.28)
\[ V = \frac{\tau_0 \omega}{\rho (f^2 - \omega^2)} \left(1 - \frac{f^2}{\omega^2} e^{-x/R_0}\right) \cos \omega t. \]  \hspace{1cm} (7.6.29)

If \( \tau_0 < 0 \) (or \( \tau_0 > 0 \)), i.e., the coast is on the right (left) of wind, the sea level near the coast rises (sinks).

**High frequency : \( \omega > f \)**

Of the two possible oscillatory solutions to (7.6.21), we must choose the one that represents outgoing waves at infinity (the radiation condition),

\[ \eta_0 = A e^{i k x}, \]  \hspace{1cm} (7.6.30)

where the wavenumber is the inverse of the modified Rossby radius of deformation,

\[ k = \sqrt{\frac{\omega^2 - f^2}{g h}} \]  \hspace{1cm} (7.6.31)

We leave it to the reader to show that, in complex form,

\[ \eta = \frac{i \tau_0 f}{\rho g h \omega k} e^{i k x - i \omega t} \]  \hspace{1cm} (7.6.32)
\[ U = -\frac{i \tau_0 f}{\rho (\omega^2 - f^2)} \left(1 - e^{i k x}\right) e^{-i \omega t} \]  \hspace{1cm} (7.6.33)
\[ V = -\frac{\tau_0}{\rho \omega \left(1 + \frac{f^2}{\omega^2 - f^2} \left(1 - e^{i k x}\right) e^{-i \omega t}\right)} \]  \hspace{1cm} (7.6.34)

or, in real form,

\[ \eta = -\frac{\tau_0 f}{\rho g h \omega k} \sin(k x - \omega t), \]  \hspace{1cm} (7.6.35)
\[ U = -\frac{\tau_0 f}{\rho(\omega^2 - f^2)} (\sin \omega t + \sin(kx - \omega t)) \]  \hspace{1cm} (7.6.36)

\[ V = -\frac{\tau_0}{\rho \omega} \left[ 1 + \frac{f^2}{\omega^2 - f^2} (\cos \omega t - \cos(kx - \omega t)) \right] \]  \hspace{1cm} (7.6.37)