8 Surface Processes

- Mass Exchange
  - Volatilization
  - Reaeration
- Momentum Transfer
  - Oil spills
- Surface Heat Transfer
  - Lake temperature models
Air water exchange

Equilibrium: Henry’s Law

\[ H = \frac{C_{ge}}{C_{le}} \]

Typical units for [H]: atm-m³/mol (\( K_H \)) or dimensionless (\( K_H' \))

For air \( K_H' \approx 42 \ K_H \)
Two-film theory

Ex: liquid side

\[ J_l = k_l (c_l - c_{le}) \]
\[ J_l = D_l \frac{(c_l - c_{le})}{z_l} \]

\[ k_l (c_l - c_{le}) = k_g (c_{ge} - c_e) = k (c_l - c_g / H) \]

\[ H = \frac{c_{ge}}{c_{le}} \]

3 eqns, 3 unknowns \((c_{le}, c_{ge}, k)\)
Two-film theory

\[ \frac{1}{k} = \frac{1}{k_1} + \frac{1}{Hk_g} \]

\[ c_{ge} - c_g = \frac{k_l (Hc_l - c_g)}{Hk_g + k_l} \]

\[ c_l - c_{le} = \frac{k_g (Hc_l - c_g)}{Hk_g + k_l} \]

Resistances in series:

- \( k_l \ll Hk_g \Rightarrow 1 \) dominates (liquid side controlled)
- \( k_l \gg Hk_g \Rightarrow 2 \) dominates (gas side controlled)

Medium with lower equilibrium concentration controls
Typical values for air and water

“Typical” values for air (as gas)

\[ D_g \sim 0.1 \text{ cm}^2/\text{s}, \ 0.1 < z_g < 1 \text{ cm} \]

\[ \Rightarrow k_g = \frac{D_g}{z_g} = 0.1 \text{ to } 1 \text{ cm/s} \]

“Typical” values for water (as liquid)

\[ D_l \sim 2 \times 10^{-5} \text{ cm}^2/\text{s}, \ 0.002 < z_l < 0.02 \text{ cm} \]

\[ \Rightarrow k_l = \frac{D_l}{z_l} = 10^{-3} \text{ to } 10^{-2} \text{ cm/s} \]

\[ K_g \sim 100 \ k_l \] so if \( H >> 0.01 \) then water side controlled (think DO); if \( H << 0.01 \) then air side controlled (think evaporation)
Example of liquid side control

\[ H \gg 0.01; \text{ assume } H \sim 1 \Rightarrow c_{ge} = c_{le} = c_e \]

\[
    c_e - c_g = \frac{k_l (Hc_l - c_g)}{Hk_g + k_l} \approx (c_l - c_g) \frac{k_l}{k_g}
\]

\[
    c_l - c_e = \frac{k_g (Hc_l - c_g)}{Hk_g + k_l} \approx (c_l - c_g)
\]
Liquid side control, cont’d

If we double $k_l$ (halve $z_l$), $(c_e - c_g)$ doubles, both gradients $\sim$ double $\Rightarrow$ twice the mass flux; red line

If we double $k_g$ (halve $z_g$), $(c_e - c_g)$ is halved, both gradients $\sim$ const $\Rightarrow$ similar mass flux; green line

Therefore mass flux controlled by liquid side
Surface Renewal Theory

Described previously for stream-reaeration formulae (Chapter 7)

- $z_l$ (or $z_w$ or $\delta$) not stagnant, but time-dependent $\sim [Dt]^{1/2}$, where $t$ is reciprocal of a renewal rate, depending on bottom generated turbulence.
- Thus $k_l$ (hence $k$) $= D_l/z_l \sim D^{1/2}$
Measurement of gas exchange

- Gas-evasion experiment: introduce chemically conservative gas (e.g., CO$_2$, propane, radon) at $c >$ saturation, and watch $c$ decline with distance due to volatilization.

- In open water bodies (or rivers where you don’t know flow rate) introduce a second, non-volatile tracer such as salt.

- Sometimes use tracer of opportunity.
Application to rivers

\[ c_{nv} = \frac{\dot{m}_{nv}}{Q_r} \]

\[ c_v = \frac{\dot{m}_v e^{-k_l x/h_u}}{Q_r} \]

\[ \frac{c_v}{c_{nv}} = \frac{\dot{m}_v}{\dot{m}_{nv}} e^{-k_l x/h_u} \]

(stream reaeration coefficient \( K_a = k_l/h \))
Gasses other than oxygen

- $K_a \sim D$ (stagnant film), $D^{1/2}$ (surface renewal), $D^{2/3}$ (split the difference)
- From Chapter 1, $Sc = \nu/D \sim MW^b$ ($b \sim 0.35$ to $0.4$)
- $K_a/K \sim (D_{O2}/D)^{2/3} \sim (32/MW)^{-1/4}$
- Example: Propane $C_3H_8$, $MW = 44$
  - $K_a/K = (32/44)^{-1/4} = 1.08$
  - Calibrations actually shows $K_a/K \sim 1.39$
How far downstream must one go?

\[ K_a = 3.9u^{0.5}h^{1.5} \]

\[ u = 0.3 \text{ m/s}, \ h = 1 \text{ m}, \ K_a = 2.1 \text{ d}^{-1} \]

\[ x \approx \frac{u}{K_a} = \frac{(0.3 \text{ m/s})(86400\text{s/d})}{(2.1\text{d}^{-1})} \approx 12 \text{ km} \]
Application to open waters

\( h = \text{water depth or thermocline depth} \)

\( \dot{m}_v \) (propane)

\( \dot{m}_{nv} \) (salt)

\[
\begin{align*}
\frac{c_v \dot{m}_{nv}}{c_{nv} \dot{m}_v} &= e^{-1} \\
0 &= 0 \\
e^{-1} &= e^{-1} \\
0 &= 0
\end{align*}
\]

\[
\begin{align*}
\dot{m}_{nv} &= \frac{m_{nv}}{\sqrt{2\pi \sigma h}} e^{-y^2/2\sigma^2} \\
c_{nv} &= \frac{\dot{m}_{nv}}{\sqrt{2\pi \sigma h}} e^{-y^2/2\sigma^2} e^{-k_x x / h} \\
c_v &= \frac{\dot{m}_v}{\sqrt{2\pi \sigma h}} e^{-y^2/2\sigma^2} e^{-k_x x / h} \\
c_v &= \frac{\dot{m}_v}{\dot{m}_{nv}} e^{-k_x x / h}
\end{align*}
\]
Mass transfer in lakes and oceans

- Most contaminants of concern are water side controlled (e.g., DO, VOC)
- In rivers, source of turbulence is bottom roughness
- In deep water bodies (lakes, oceans) it is wind stress $\Rightarrow u_{w*}$ (water-side friction velocity) which affects $z_l$
- Contaminants that are air side controlled also affected by wind (through $z_g$)
$k_l$ vs $u_{w^*}$

$u_{w^*}$ because transfer is water side controlled and $u_{w^*}$ is indicator of turbulence; yet $u_{w^*}$ not easily measured.
Wind Stress

\[ \tau = C_{10} \rho_a u_{10}^2 = \rho_w u_{w*}^2 \]

\[ u_{w*} = \sqrt{\frac{C_{10} \rho_a}{\rho_w}} u_{10} \]

\( u_{10} = 10 \text{ m wind speed}; \)

\( C_{10} = \text{drag coef.} \)

\( C_{10} = (0.8+0.065u_{10}) \times 10^{-3} \)

\[ [u_{10} > 1 \text{ m/s}; \text{Wu, 1980}] \]

\( u_{10} \rightarrow C_{10} \rightarrow u_{w*} \rightarrow k_l \)
$k_l$ (or $z_l$) vs $u_{10}$

Yu and Hamrick (1984)  
Emerson (1075)

**Graph:***

- **$K_l$, Overall Liquid Film Coefficient at 20°C, m/sec.**
- **$U_{10}$, Average Wind Speed, m/sec. at 10 m**
- **$n = 1$**, **$n = 2$**, **$n = 3/2$**, **$n = 3/4$**

**Legend:**
- Downing and Truesdale
- Juliano
- Mattingly
- Yu et al

**Data Points:**
- Konwisher (1963) Smooth surface
- Konwisher (1963) 3 cm waves 60/mm
- Hoover and Berkshire (1969)
- Thurber and Broecker (1970)
- Liss (1973)

**Lines:**
- ELA Lakes
- Central, Atlantic (Broecker and Peng 1971)
- North Pacific (Peng et al. 1974)

**Figures by MIT OCW.**
Example film coefficients

\[ k_l = 0.0004 + 0.00004u_{10}^2 \]

\[ k_g = 0.3 + 0.2u_{10} \]

\(k_l\) and \(k_g\) in cm/s; \(u_{10}\) in m/s [Schwarzenbach et al, 1993]

Note that both depend on \(u_{10}\)
Examples

Above eqns:

\[ u_{10} = 5 \text{ m/s} \Rightarrow k_l = 1.4 \times 10^{-3} \text{ cm/s (green dot)}; \]
\[ k_g = 1.3 \text{ cm/s} \]

Figure 8.8:

\[ z_l = \delta = 120 \mu \text{m} = 1.2 \times 10^{-2} \text{ cm}. \]
For DO, \( D = 2 \times 10^{-5} \text{ cm}^2/\text{s} \)
\[ k_l = D/z_l = 2 \times 10^{-5}/1.2 \times 10^{-3} = 1.7 \times 10^{-3} \text{ cm/s (red dot)} \]
$k_1$ (or $z_1$) vs $u_{10}$

Yu and Hamrick (1984)

Figure by MIT OCW.

Emerson (1075)
Volatile Halogenated Organic Compound (VHOC) Experiment

- CH$_3$Cl$_3$ and other one carbon VOCs (THMs) and two carbon VOCs (solvents) discharged with waste water.

- Used to
  - compute volatilization (assuming known residence time) or
  - compute residence time (with known volatilization)
TCE data in Boston Harbor

TCE loading from Deer & Nut Island TPs: 24 m³/s at 11 µg/L

Ave harbor TCE concentration
241 ng/L (all pts)
214 ng/L (excl. 840)

Harbor volume = 6x10⁸ m²

Kossik, Gschwend & Adams, 1987
TCE Experiment, cont’d

Nominal residence time (w/o volatilization; excluding presumed outlier)

\[ \tau^* = \frac{\bar{c} V}{Q_o c_o} = \frac{(214 \times 10^{-9} \text{ kg} / \text{m}^3)(6 \times 10^8 \text{ m}^3)}{(24 \text{ m}^3 / \text{s})(86400 \text{ s} / \text{d})(11 \times 10^{-6} \text{ kg} / \text{m}^3)} = 5.6d \]
TCE Experiment, cont’d

With volatilization

\[ V \frac{dc}{dt} = -kAc - Q_f \]

\[ \frac{dc}{dt} = -\left( \frac{kA}{V} + \frac{Q}{V} \right)c \]

\[ \kappa^* = 1/\tau^* \]

For \( \text{CH}_3\text{Cl}_3 \) \( H = 1.13 \) (dimensionless) \( >> 1 \) \( \Rightarrow \) ws control

\[ D = 1.0 \times 10^{-5} \, \text{cm}^2/\text{s} \]
From Figure 8.8 and $u_{10} = 5$ m/s, $\delta = 1.2 \times 10^{-2}$ cm.

\[
k = \frac{D}{\delta} = \frac{1.0 \times 10^{-5}}{1.2 \times 10^{-2}} = 0.00083 \text{ cm/s} = 3 \text{ cm/hr} = 0.72 \text{ m/d}
\]

\[
\frac{1}{t^*} = \frac{1}{t} + \frac{k}{h}
\]

\[
\frac{1}{t} = \frac{1}{t^*} - \frac{k}{h} = \left(\frac{1}{5.6d}\right) - \frac{(0.72 \text{ m/d})}{6 \text{ m}}
\]

\[
= 0.18 - 0.12 = 0.06 \text{ d}^{-1} \Rightarrow \tau = 17 \text{ d}
\]

Estimated $\tau$ is too high; reason is likely extraneous or under-accounted sources of CH$_3$Cl$_3$. 
Momentum Exchange

-Chapters 2, 3 discussed surface shear stress for eddy diffusivity and hydrodynamic modeling

-Previous section discussed stress as source of turbulence governing mass exchange

-Also of interest in transporting floating material, specifically spilled hydrocarbons
Oil Spills

- Composition
- Fate
- Transport (spreading, advection)
- Clean-up
 Marine Sources \((10^3 \text{ MT/yr})\)

<table>
<thead>
<tr>
<th>Source</th>
<th>N. America</th>
<th>Global</th>
</tr>
</thead>
<tbody>
<tr>
<td>Natural Seeps</td>
<td>160</td>
<td>600</td>
</tr>
<tr>
<td>Petroleum Extraction</td>
<td>3</td>
<td>38</td>
</tr>
<tr>
<td>Petroleum Transport</td>
<td>9</td>
<td>150</td>
</tr>
<tr>
<td>Petroleum Consumption</td>
<td>84</td>
<td>480</td>
</tr>
<tr>
<td>Total</td>
<td>260</td>
<td>1300</td>
</tr>
</tbody>
</table>

About half is anthropogenic

\((\text{Oil in the Sea III, NRC, 2003})\)
Composition

- Crude and Refined Oils
- Always multiple constituents
- Characterized by Boiling Point (or distillation cut)
Fate

- Volatilization (lighter fractions)
- Emulsification (depending on oil)
- Natural dispersion (if enough energy)
- Biodegradation
- Dissolution
- Photo-oxidation
- Sediment particle interaction

Output from NOAA’s ADIOIS model; independent of transport
Transport Models

- Spreading and Advection
- Pre-planning (evaluate risk)
- Real-time (assist clean-up; needs to be quick and dirty)
- Hind-cast (who is responsible, damage assessment)
Simple advection model

\[
\mathbf{\tau}_a = \frac{f_a}{2} \rho_a u_a^2 = \frac{f_w}{2} \rho_w (u_s - \bar{u}_w)^2 = \frac{f_w}{2} \rho_w (\Delta u_s)^2
\]

\[
f_w \approx f_a \quad \Rightarrow \quad \frac{\Delta u_s}{u_a} = \sqrt{\frac{\rho_a}{\rho_w}} \approx 0.03
\]

Surface current speed \( \sim 3\% \) of wind speed. (Also explained by Stokes Drift due to surface waves)

In which direction?
Ekman Model

Linearized equations of motion; constant viscosity

\[ \frac{\partial u}{\partial t} - \Omega v = E \frac{\partial^2 u}{\partial z^2} \]

\[ \frac{\partial v}{\partial t} + \Omega u = E \frac{\partial^2 v}{\partial z^2} \]

\[ w = 0 \]

\[ E \frac{\partial w}{\partial z} = \frac{\tau_{sx} + i \tau_{sy}}{\rho_w} \]

\[ \Omega = 2\omega \sin \phi \]

Coriolis parameter

Complex velocity

At depth \((z = -\infty)\)

At surface \((z = 0)\)
Ekman Model, cont’d

\[ w = \frac{\tau_{sy}}{\rho_w \sqrt{E\Omega}} \exp\left\{ \sqrt{\frac{\Omega}{2E}} (1 + i)z + \frac{i\pi}{4} \right\} \]

Surface drift 45° to right;
Depth average drift 90° to right
Field experiments show surface drift \( \sim 10° \) to right. Explained by variable vertical viscosity \( E \sim z \) (Madsen, 1977)
Other effects of wind: Coastal Upwelling/Downwelling
Other effects of wind: Langmuir Circulation

Figure by MIT OCW.
Idealized Spreading (Fay, 1969)

\[
\begin{align*}
\frac{dD}{dt} & \sim \sqrt{g'h} \sim \frac{1}{D} \sqrt{g'V} \\
\frac{dD}{dt} & \sim \frac{g'V^2}{D^5} \sqrt{\frac{t}{\nu_w}} \\
\frac{dD}{dt} & \sim \frac{f_r}{\rho_w D} \sqrt{\frac{t}{\nu_w}} \\
\end{align*}
\]

Gravity-Inertia
Gravity-Viscous
Surface Tension-Viscous
### Idealized spreading, cont’d

<table>
<thead>
<tr>
<th>Regime</th>
<th>Formula</th>
<th>$K_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gravity-Inertia</td>
<td>$D = 2k_1[g'Vt^2]^{1/4}$</td>
<td>$K_1 = 1.14$</td>
</tr>
<tr>
<td>Gravity-Viscous</td>
<td>$D = 2k_2[g'V^2t^{3/2}/\nu_w^{1/2}]^{1/6}$</td>
<td>$K_2 = 0.98$ to $1.45$</td>
</tr>
<tr>
<td>Surface Tension-Viscous</td>
<td>$D = 2k_3[f,t^3/\rho_w^2\nu_w]^{1/4}$</td>
<td>$K_3 = 1.6$</td>
</tr>
</tbody>
</table>
Comments

- Theory applies down to slick thickness of about 0.1 mm
- Additional spreading due to
  - Time-varying spillage
  - Wind, waves and non-uniform currents
  - Dispersion of submerged (slower moving) oil droplets
- Field experiments show oil often very non-uniform (90% of volume in 10% of area)
Oil Transport Models

- Slick advected with underlying surface current plus 3% of wind speed (~10% deflection to right)
- (3-D) models simulate transport of subsurface dispersed oil.
- Currents can be observed or predicted (sophistication depends on application—available time)
- Fate processes often computed independently from transport
Model Simulations

NOAA’s 3D GNOME; ANS Crude off Coast of Florida
Mechanical Clean-up
Chemical Dispersion

- Surfactants that reduce interfacial tension
- Create dispersed droplets
- Subsurface/bottom impacts vs surface/shoreline
- Air (large spills) or boat application
- Window of opportunity
Chemical Dispersion, cont’d

Application

Hydrophilic Group

Lipophilic Group

Water

Hydrophil
Portion of Dispersant Prevents Droplet Coalescence

Surfactant-Stabilized Oil Droplets

NRC, 1989
In situ Burning

- Considered secondary option (like chemical dispersants)
- Most appropriate for offshore spills (reduced AQ impacts)
Surface Heat Transfer and Temperature Modeling

- Surface heat fluxes
- Linearized surface heat transfer
- Cooling ponds
- Natural lakes and reservoirs
Importance of Temperature

- Important WQ parameter
  - Thermal pollution
  - Species preference (fish habitat)

- Affects rate constants
  - $K = K_{20} \theta^{T-20}$

- Produces density stratification
  - $\rho = \rho(T)$

- Important tracer (e.g., $E_z$)
Surface Heat Transfer (W-m\(^{-2}\))

\[
\phi_n = \phi_{sn} + \phi_{an} - \phi_{br} - \phi_e - \phi_c
\]

<table>
<thead>
<tr>
<th>Heat Source</th>
<th>Value Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Net solar, (\phi_{sn})</td>
<td>60 to 300</td>
</tr>
<tr>
<td>Net atmospheric, (\phi_{an})</td>
<td>200 to 450</td>
</tr>
<tr>
<td>Back radiation, (\phi_r)</td>
<td>250 to 500</td>
</tr>
<tr>
<td>Evaporation, (\phi_e)</td>
<td>0 to 350</td>
</tr>
<tr>
<td>“Conduction”, (\phi_c)</td>
<td>-70 to 200</td>
</tr>
</tbody>
</table>
Solar Radiation

- Short wave length (< 3µm)
- Direct plus diffuse (scattered, reflected)
- Absorbed & re-radiated (> 3µm) by clouds
- Measured by pyranometer
- Incident clear sky radiation calculated from latitude, date and time of day
- Corrections for cloud cover and reflection
Net Solar Radiation (cont’d)

<table>
<thead>
<tr>
<th>$\phi_{sr}/\phi_s$ (%)</th>
<th>Jan</th>
<th>Feb</th>
<th>Mar</th>
<th>Apr</th>
<th>May</th>
<th>Jun</th>
<th>Jul</th>
<th>Aug</th>
<th>Sep</th>
<th>Oct</th>
<th>Nov</th>
<th>Dec</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>9</td>
<td>7</td>
<td>7</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>6</td>
<td>7</td>
<td>9</td>
<td>10</td>
</tr>
</tbody>
</table>

$$\phi_{sn} = \phi_s - \phi_{sr} \equiv 0.94\phi_{sc} \left(1 - 0.65C^2\right)$$

C = fractional cloud cover
Depth-variation of solar radiation

Measured with Secchi disk or *in-situ* pyranometer

\[ \phi_z = (1 - \beta) \phi_{sn} e^{-\eta z} \]

\[ \eta = \frac{1.7}{d_D} \]

\[ \beta \approx 0.5 \]
Atmospheric Radiation

- Long wave length (> 3\(\mu\)m)
- Re-radiated from atmosphere
- Measured by pyrgeometer
- Incident clear sky radiation calculated from absolute air temperature, vapor pressure
- Corrections for cloud cover and reflection
Incident Radiation Formulae

\[ \phi_{ac} = \varepsilon \sigma (T_a + 273)^4 \]

\( \sigma = \text{Stefan-Boltzmann constant } (5.7 \times 10^{-8} \text{ W/m}^2\cdot\text{oK}^4) \)

\( \varepsilon = \text{emissivity (dimensionless)} \)

\[ \varepsilon = 0.92 \times 10^{-5} (T_a + 273)^2 \] \text{Swinbank (1963)}

\[ \varepsilon = \left\{ 1.0 - 0.26 / \exp [7.77 \times 10^{-5} (T_a)^2] \right\} \] \text{Itso-Jackson (1969)}

\[ \varepsilon = 1.24 \left( \frac{e}{(T_a + 273)} \right)^{1/7} \] \text{e = vapor pressure, mbar Brutsaert (1975)}
Net Atmospheric radiation

\[ \phi_{an} = 0.97 \varepsilon \sigma (T_a + 273)^4 \left( 1.0 + 0.17 C^2 \right) \]

C = fractional cloud cover

\sim 3\% \ reflection
Back Radiation

Water surface is nearly a black body

\( \varepsilon \approx 0.97 \)

\[
\phi_{br} = 0.97 \sigma (T_s + 273)^4 = 5.5 \times 10^{-8} (T_s + 273)^4
\]
Evaporative Heat Flux

- Measured
  - eddy flux (short term)
  - evaporation pans (long term)
- Computed from mass transfer formulae

\[ E = \rho f(W_z) (e_s - e_z) \]

- \( e_s \): vapor pressure at surface
- \( e_z \): vapor pressure at elevation \( z \)
- \( f(W_z) \): wind speed function = \( a + bW_z \) (\( k_g \))

Dalton’s Law
Evaporative Heat Flux (cont’d)

Mass transfer => heat transfer using latent heat of vaporization

\[ L_v = (2493 - 2.26T_s) \times 10^3 \, \text{J/Kg} \]

\[ \phi_e = L_v E = f(W_z)(e_s - e_a) \]

\[ \phi_e = 3.72W_2(e_s - e_2) \]

\[ \phi_e = 5.1A^{-0.05}W_2(e_s - e_2) \]

(W/m²; W₂ in m/s; eₛ, e₂ in mb)

"Lake Hefner", Marciano and Harbeck (1954)

"Fetch-dependence” Harbeck, (1962)
$e_z$ and $W_z$ vary vertically (height above water) and horizontally (above water or on-shore)
Evaporation from non-natural water bodies

- $e_s$ increases with temperature
  - Heated water bodies have increased evaporation (water vapor also lighter than air)
- $e_s$ decreases with salinity
  - Saline bodies have decreased evaporation
- $e_s$ decreases with pressure

\[ \phi_e = f(W_z) \left( e_s - e_z \right) \]
Conductive Heat Flux

Computed from evaporative flux using Bowen Ratio

\[ \phi_c = R_b \phi_e \]

\[ R_b = C_b \frac{(T_s - T_z)}{(e_s - e_z)} \]

\[ C_b = 0.61 \text{ mb/}^\circ\text{C}; \]
Summary

\[ \phi_n = \phi_{sn} + \phi_{an} - \phi_{br} - \phi_e - \phi_c \]

functions of external factors (met and astronomical conditions)

functions of \( T_s \)

Strategies for computation: table look up

Self regulation: errors in calculations compensate
Linear Heat Transfer

Equilibrium Temp, $T_e$
- $T_s$ for which $\phi_n = 0$
- Function of met

Surface Heat Exchange Coefficient, $K$
- Slope of $\phi_n$ vs $T_s$

$\phi_n = -K(T_s - T_e)$

$K \sim 20-50$ W/m$^2$°C
Example: Periodic Heat Loss

\[ \rho c V \frac{dT}{dt} = A_s \phi_n \]

\[ \frac{dT}{dt} = -k(T - T_e) \quad k = K/\rho c_p h \]

\[ T_e = \bar{T}_e + \Delta T e^{i\omega t} \quad \omega = 2\pi/P \]

\[ T = \bar{T} + \Delta T * e^{i\omega t} \]

\[ T = \bar{T}_e + \Delta T e^{i\theta} e^{i\omega t} \]
Periodic Heat Loss (cont’d)

\[ T = \bar{T} + \Delta T^* e^{i\omega t} \quad \Delta T^* = \Delta T e^{i\theta} \]

\[ T = \bar{T}_e + \Delta T e^{i\theta} e^{i\omega t} \]

\[ T = \bar{T}_e + \Delta T e^{i\omega (t-t_L)} \quad t_L = (\theta/2\pi)P \]

\[ \frac{\Delta T^*}{\Delta T_e} = \frac{k}{k + i\omega} = \frac{k}{\sqrt{k^2 + \omega^2}} e^{i\tan^{-1}(-\omega/k)} \]

Phase lag

Amplitude

\[ \theta = \tan^{-1}(-\omega/k) \]

\[ t_L = \frac{P}{2\pi} \tan^{-1}(\omega/k) \]
**Examples**

\[ \frac{K}{\rho c} = 1 \text{m/d}^*; \ h = 10 \text{m}, \ k = \frac{K}{\rho c} h = 0.1 \text{d}^{-1} \]

<table>
<thead>
<tr>
<th>(P)</th>
<th>1 day</th>
<th>365</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\omega = \frac{2\pi}{P})</td>
<td>0.17(d^{-1})</td>
<td>6.28(d^{-1})</td>
</tr>
<tr>
<td>(\Delta T/\Delta T_e = \frac{k}{(k^2 + \omega^2)^{0.5}})</td>
<td>0.016</td>
<td>0.986</td>
</tr>
<tr>
<td>(\theta = \tan^{-1}(-\omega/k))</td>
<td>-89°</td>
<td>-10°</td>
</tr>
<tr>
<td>(t_L = \frac{P}{2\pi \tan^{-1}(\omega/k)})</td>
<td>0.247 d</td>
<td>10 d</td>
</tr>
</tbody>
</table>

* \(K \sim 48 \text{ W/m}^{2\circ}C\)
Cooling Lakes and Ponds

- Used to cool electric power plants
- Shallow (vertically well-mixed)
  - Erected with dikes
  - $T = T(x,y) + T(t)$
- Deep reservoirs
  - Damming of reservoirs
- Cooling capacity
  - $r = K A_p / \rho c Q_o$
Example: shallow-longitudinal dispersive

\[ \rho c Q_o \frac{dT}{dx} = \rho c W H E_L \frac{d^2 T}{dx^2} - K(T - T_E)W \]

Single pass

\[ \frac{T_i - T_E}{T_o - T_E} = \frac{4ae^{1/2E_L^*}}{(1 + a)^2 e^{a/2E_L^*} - (1 - a)^2 e^{-a/2E_L^*}} \]

Continuous operation

\( (T_o = T_i + \Delta T_o) \)

\[ \frac{T_i - T_E}{\Delta T_o} = \frac{4ae^{1/2E_L^*}}{(1 + a)^2 e^{a/2E_L^*} - (1 - a)^2 e^{-a/2E_L^*} - 4ae^{1/2E_L^*}} \]

Figure by MIT OCW.

Jirka et al. (1978)
Stratification in Lakes & Reservoirs

- Factors causing vertical stratification
  - Differential absorption
  - Reduced vertical mixing

- Factors causing horizontal stratification
  - Strong through flow
  - Strong wind
  - Differential absorption
Reservoir classification based on horizontal through flow (Orlob, 1969)

- Through flow velocity = \( \frac{L}{(V/Q)} \)
- Int’l wave speed \( \sim \) \((g\Delta \rho/\rho h)^{0.5} \sim Nh\)
  - \( N = \) buoyancy freq = \( [(g/\rho)(d\rho/dz)]^{0.5} \)
  - \( L = \) length; \( Q = \) flow; \( h = \) depth; \( V = \) vol
- \( Fr = \frac{LQ}{VNh} \)
  - \( Fr << 1 \) vertically stratified
  - \( Fr >> 1 \) vertically mixed
1-D Reservoir Modeling

\[ \frac{\partial Q}{\partial z} = q_{\text{in}} - q_{\text{out}} \]

\[ \frac{\partial T}{\partial t} + \frac{1}{A} \frac{\partial}{\partial z} (QT) = \frac{1}{A} \frac{\partial}{\partial z} \left[ AE_z \frac{\partial T}{\partial z} \right] + \frac{q_{\text{in}} T_{\text{in}} - q_{\text{out}} T}{A} \]
Surface Layer

- Well mixed layer
  - Convective mixing
  - Wind mixing
- Wind mixing algorithm for surface
  - Oceans (Kraus-Turner)
- 1-D model below
Surface Layer (cont’d)

\[
\frac{\Delta PE}{A} = (\Delta \rho g \Delta h) \frac{h}{2} = \frac{\Delta \rho gh}{2} u_e \Delta t
\]

\[
\frac{\Delta KE}{A} = \rho u_*^2 u_s \Delta t \sim \rho u_*^3 \Delta t
\]

\[
\frac{\Delta PE}{\Delta KE} = \text{const}
\]

\[
u_e \sim \frac{u_*^2}{u_*} = \frac{\Delta \rho}{\rho} \frac{gh}{\rho}
\]

\[
\tau = \rho u_*^2
\]

Many variants
Lake stability

Stability index (PE of water body with equivalent mass and heat content but uniform density – PE of stratified body)

\[ SI = \int_0^h \left[ \bar{\rho} - \rho(z) \right][z - z_c]gA(z)dz \]

\[ \bar{\rho} = \frac{\int_0^h \rho(z)A(z)dz}{\int_0^h A(z)dz} \quad \text{Average density} \]

\[ z_c = \frac{\int_0^h \rho(z)A(z)zdz}{\int_0^h \rho(z)A(z)dz} \quad \text{Center of mass} \]