Stress and strain from a screw dislocation

Need to get traction = 0 at surface. First consider infinite medium.

Assume:

\[ u_1 = u_3 = 0 \]
\[ u_2 = \frac{S \theta}{2\pi} \]

\[ \nabla \cdot u = 0 \Rightarrow \text{no compression, only shear} \]

Symmetry \( \Rightarrow \) cylindrical coordinates, \( r, \theta, z \)

with \( z \) parallel to \( x_2 \) axis; \( u_r = u_\theta = 0 \)

Solution is \( \sigma_{\theta z} = \frac{\mu \partial u_z}{r \partial \theta} = \frac{\mu S}{2\pi r} \)

We can get \( \sigma_\theta \) by coordinate transformation.
How to get traction on surface?

Trick – image dislocation

Solution for matched image dislocation is whole space gives $\sigma_{i3} = 0$ on surface of $\frac{1}{2}$ space!

Shear strain at surface:
\[ r^2 = x_1^2 + x_3^2 \]

From each dislocation \( \varepsilon_{z\theta} = \frac{S}{2\pi r} \)

Rotating strain tensor
\[ \varepsilon_{12} = \varepsilon_{z\theta} \cos \theta = \varepsilon_{z\theta} \frac{x_3}{r} \]

At surface
\[
\varepsilon_{12}^D = \frac{S}{2\pi} \left[ \frac{x_3}{x_1^2 + x_3^2} + \frac{2D - x_3}{(2D - x_3)^2 + x_1^2} \right] = \frac{SD}{\pi(D^2 + x_1^2)}
\]

where
\[
\frac{x_3}{x_1^2 + x_3^2} \quad \text{is actual dislocation}
\]
\[
\frac{2D - x_3}{(2D - x_3)^2 + x_1^2} \quad \text{is image dislocation}
\]

Displacement
\[
u_2 = \int_0^\infty \varepsilon_{12}^D dx_1 = \frac{S}{2} \left( 1 - \frac{2}{\pi} \tan^{-1} \frac{x_1}{D} \right)
\]

Aside – slip discontinuity objectionable?
\[
\sigma_{12} \to \infty \quad \text{along } x_1 = 0 \text{ as } x_3 \to 0
\]
(stress singularity at tip of fault)

Alternative model
Apply uniform \( \sigma_{12}^0 \)
Cut \( 0 \leq x_3 \leq D \), set \( \sigma_{12} = 0 \)
\[ u_2 = \frac{S}{2} \left( \left( 1 + \frac{x_1^2}{D^2} \right)^{1/2} - \frac{x_1}{D} \right) \]

Figure 19.4
Figure by MIT OCW.

Virtually indistinguishable!

**St. Venant’s principle**

Elastostatics – if boundary tractions on a part \( S_1 \) of the boundary of \( S \) are replaced by statically equivalent traction dist, effects on stress dist are negligible at pts whose distance from \( S_1 \) is large compare to size of \( S \).
Usual context – long beam under end load (non-uniform)

![Diagram of a long beam under end load](image)

**Figure 19.5**

Figure by MIT OCW.

Apply to loading 1/2 space

![Diagram of a rock being applied to a line](image)

**Figure 19.6**

Pt source approximates in seismology.