Chapter 2

Deep water gravity waves

2.1 Surface motions on water of finite depth

We now move on to consider motions in water that is not shallow, *i.e.*, we will allow the flow to vary with z within the water:

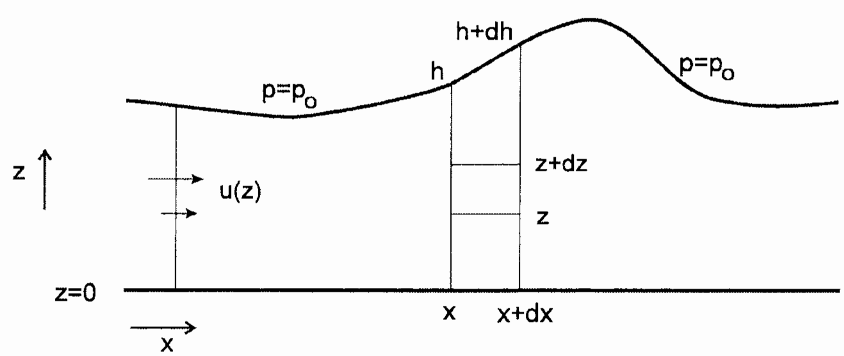

Figure 2.1: Configuration of deep water system.

As before, the system is nonrotating and inviscid, and the water density ρ is assumed constant. Basically, we follow the same procedures as for the shallow water case. However, rather than consider balances for columns of water we must do so separately for elemental volumes $dx\,dz$, such as the box in the figure. The horizontal and vertical momentum equations are[1]

$$
\begin{aligned}
\rho \frac{du}{dt} &= \rho \frac{\partial u}{\partial t} + \rho u \frac{\partial u}{\partial x} + \rho w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x} \; ; \\
\rho \frac{dw}{dt} &= \rho \frac{\partial w}{\partial t} + \rho u \frac{\partial w}{\partial x} + \rho w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} - g\rho \; .
\end{aligned}
\tag{2.1}
$$

Thus, the processes tending to accelerate the flow in the x-direction are: the x-gradient of pressure and advection of x-momentum; in the z-direction: the z-gradient of pressure, gravity, and advection of z-momentum.

We now consider conservation of mass within the marked volume; its mass, per unit length in y, is $m = \rho \, dx \, dz$, which for this **incompressible** medium is constant with time.

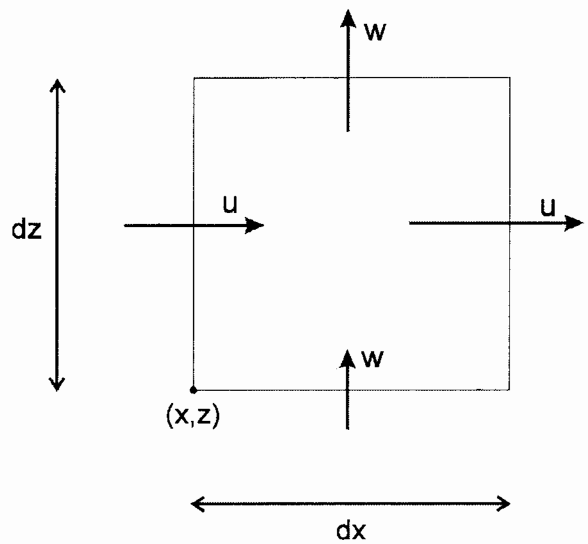

Figure 2.2: Mass continuity.

However, there are fluxes of mass across the volume, as shown in Fig. 2.2. Across the left face, into the box, is a mass flux, per unit length in y, of $\rho \, u(x,z) \, dz$; there is an outward flux of $\rho \, u(x+dx, z) \, dz$ across the right face. Similarly, there is a flux $\rho \, w(x,z) \, dx$ in through the lower face and

[1]Since the varaibles are now functions of (x, z, t), the substantive derivative is now $\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dz}{dt}\frac{\partial}{\partial z}$.

a flux $\rho\ w(x, z + dz)\ dx$ out through the upper face. Altogether, the net **convergence** of the mass fluxes into the box is

$$\begin{aligned} C &= \rho\, dz\, [u(x,z) - u(x+dx,z)] + \rho\, dz\, [w(x,z) - w(x,z+dz)] \\ &= -\rho\, dx\, dz \left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) = -\rho\, dx\, dz\, \nabla \cdot \mathbf{u} \end{aligned}$$

where $\mathbf{u}$ is the vector velocity (u, w). Continuity of mass demands, since there is no change of mass within the box, that C be zero, whence our **continuity equation**

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0\ . \tag{2.2}$$

This tells us that incompressible flow is **nondivergent**.

So we have 3 equations, the 2 momentum eqs. (2.1) and the continuity eq. (2.2), in three unknowns u, w, and p. We also have boundary conditions. At the lower boundary $z = 0$, there can be no normal motion, whence $w = 0$ there[2]. The motion will in general be nonzero at the free surface, since this can move. However, fluid on the surface (which is a **material surface**) must remain there—it cannot pass through, which tells us that a fluid parcel on the surface must move with the local fluid speed along the surface, or

$$\frac{dh}{dt} = w\big|_{z=h}\ . \tag{2.3}$$

Finally, we also know that the pressure in the air immediately above the surface is atmospheric pressure p_0 which, as before, we assume constant. Since pressure must be continuous across the surface (to be otherwise would imply an infinite pressure gradient and therefore infinite acceleration, which would be unphysical[3]). So our final boundary condition is

$$p\big|_{z=h} = p_0\ . \tag{2.4}$$

[2]Note that we can impose no similar condition on u at the lower boundary, since we have assumed inviscid flow and so cannot include viscous boundary effects.

[3]We are in fact neglecting surface tension here. If the surface is curved, as it will be in the presence of the wave, surface tension will exert an effective pressure on the fluid beneath. Such effects are significant only for waves of wavelength a few cm or less (and such waves are known as **capillary waves**).

2.2 Small-amplitude deep-water surface waves

As before, we investigate the properties of small-amplitude disturbances to a stationary basic state, described by

$$\begin{aligned} u &= w = 0 \,, \\ h &= D \,, \\ p &= p_0 + g\rho(D - z) \,. \end{aligned} \tag{2.5}$$

Introducing small perturbations u', w', h', and p', the momentum eqs. (2.1) can then be written as

$$\begin{aligned} \frac{\partial u'}{\partial t} + \frac{1}{\rho}\frac{\partial p'}{\partial x} &= -u'\frac{\partial u'}{\partial x} - w'\frac{\partial u'}{\partial z} \,, \\ \frac{\partial w'}{\partial t} + \frac{1}{\rho}\frac{\partial p'}{\partial z} &= -u'\frac{\partial w'}{\partial x} - w'\frac{\partial w'}{\partial z} \,. \end{aligned}$$

Now, the terms in the RHS are *nonlinear*, in fact quadratic in the perturbation quantities; we neglect these on the grounds that the perturbations are small, leaving

$$\frac{\partial u'}{\partial t} + \frac{1}{\rho}\frac{\partial p'}{\partial x} = 0 \,, \tag{2.6}$$

$$\frac{\partial w'}{\partial t} + \frac{1}{\rho}\frac{\partial p'}{\partial z} = 0 \,. \tag{2.7}$$

The continuity eq. (2.2) becomes simply

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0 \,. \tag{2.8}$$

We now proceed to derive a single equation in a single unknown from these three. In fact, it is very simple to do so: taking the x-derivative of (2.6), the z-derivative of (2.7), and adding gives

$$\frac{\partial}{\partial t}\left(\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z}\right) + \frac{1}{\rho}\left(\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial z^2}\right) = 0 \,.$$

But, from (2.8), the term in the first parenthesis is zero, so

$$\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial z^2} = 0 \,. \tag{2.9}$$

As before, we focus attention on wave-like disturbances, for which

$$p'(x,z,t) = Re\left[\widetilde{P}(z,t)e^{ikx}\right] ; \tag{2.10}$$

substitution into (2.9) then gives

$$\frac{\partial^2 \widetilde{P}}{\partial z^2} - k^2 \widetilde{P} = 0 ,$$

which has solutions of the form

$$\widetilde{P}(z,t) = P_1(t)e^{kz} + P_2(t)e^{-kz} . \tag{2.11}$$

(Note that this tells us that the vertical scale of the motion is k^{-1}, which is formally the same as the horizontal length scale.) Now, the lower boundary condition tells us that $w' = 0$ at $z = 0$; therefore, from (2.7), $\partial p'/\partial z = 0$ there. So, using (2.10), we have

$$\frac{\partial \widetilde{P}}{\partial z}(0,t) = 0$$

whence, in (2.11), $P_1 = P_2 = P(t)/2$, say, and so (2.10) becomes

$$p'(x,z,t) = Re\left[P(t)\ \cosh kz\ e^{ikx}\right] . \tag{2.12}$$

It is now straightforward, from (2.6) and (2.7), to obtain the form of the velocity perturbations

$$u'(x,z,t) = Re\left[-i\frac{k}{\rho}Q(t)\ \cosh kz\ e^{ikx}\right] , \tag{2.13}$$

$$w'(x,z,t) = Re\left[-\frac{k}{\rho}Q(t)\ \sinh kz\ e^{ikx}\right] , \tag{2.14}$$

where $dQ/dt = P$.

Notice that, at this stage we have defined the spatial structure of the motions, but not the time dependence. To define the latter, we have to include consideration of the surface dynamics.

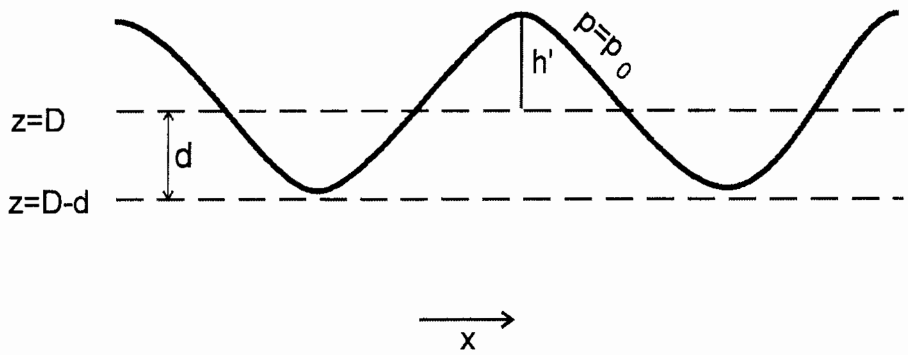

Figure 2.3: Near-surface details.

Consider the level $z = D - d$, where d is just greater than $|h'|$, so the level lies just beneath the wave troughs, as shown in Fig. 2.3. (Since our equations are valid only within the water, we cannot choose $z = D$.) Now, since [eq. (2.4)] the surface pressure is p_0, the total pressure (background + perturbation) is, using hydrostatic balance,

$$p(x, D - d, t) = p_0 + g\rho(d + h') .$$

But, since the basic state pressure is, from (2.5), $p_0 + g\rho d$, the perturbation is $p'(x, D - d, t) = g\rho h'$; taking the limit $d \to 0$ (recall that h' is arbitrarily small), we have

$$p'(x, D, t) = g\rho h' , \tag{2.15}$$

so, from (2.12),

$$h'(x, t) = \frac{1}{g\rho} Re \left[P(t) \ \cosh kD \ e^{ikx} \right] . \tag{2.16}$$

(Note that $p'(x, D, t)$ is nonzero—it is the pressure perturbation *at the surface* $z = D + h'$ that is zero.) The material surface condition (2.3), neglecting nonlinear terms, is

$$\frac{\partial h'}{\partial t} = w'(x, D, t) .$$

Using (2.15),

$$\frac{\partial p'}{\partial t}(x, D, t) = g\rho w'(x, D, t) .$$

Substituting from (2.12) and (2.14), we get

$$\frac{dP}{dt}(t)\ \cosh kD = -gk\ Q(t)\ \sinh kD$$

or, since $P = dQ/dt$,

$$\frac{d^2Q}{dt^2} + gk\ \tanh kD\ Q = 0\ . \tag{2.17}$$

This has solutions

$$Q = Q_+ e^{-i\omega t} + Q_- e^{i\omega t} \tag{2.18}$$

provided ω satisfies the dispersion relation

$$\omega^2 = gk\ \tanh kD\ . \tag{2.19}$$

Eq. (2.19) can be rewritten as

$$\omega = \pm\frac{c_0}{D}\sqrt{kD\ \tanh kD}\ , \tag{2.20}$$

where $c_0 = \sqrt{gD}$ is the shallow water wave speed. (2.20) is plotted in Fig. 2.4.

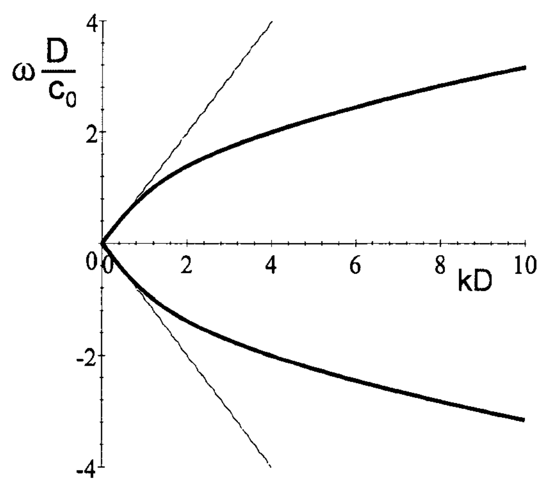

Figure 2.4: Dispersion relation for deep water gravity waves. Dashed lines show the shallow water relation.

Note the departure from shallow water behavior for $kD > 1$. In general, the most important difference is that, for deep water waves, the phase speed $c = \omega/k$ is *not* independent of wavenumber—such waves are known as **dispersive waves**.

2.2.1 The long wave (shallow water) limit

Note from (2.20) that, in the limit $kD \to 0$, $\omega^2 \to gDk^2$, so we recover the shallow water dispersion relation (and, in fact, shallow water dynamics) in this limit when the wavelength is much greater than the depth. However, kD in fact has to be *very* small for this approximation to be valid: the wavelength $2\pi/k$ must be greater than 14 times the depth before the shallow water result for c becomes within 3% of the correct value.

From (2.13), since $\cosh kz \to 1$ as $kD \to 0$ (note that $z \leq D$) the horizontal velocity becomes independent of z in this limit, as we assumed in our shallow water analysis.

2.2.2 The short wave (deep water) limit

For $kD \to \infty$, $\tanh kD \to 1$, so (2.19) becomes

$$\omega = \pm\sqrt{gk} \tag{2.21}$$

which is independent of D. In fact, the whole problem becomes insensitive to the background depth in this limit, as the waves do not feel the bottom. The vertical structure functions $\cosh kz$ and $\sinh kz$ appearing in (2.13), (2.14) and (2.12) are

$$\begin{aligned}\cosh kz &= \frac{1}{2}\left(e^{kz} + e^{-kz}\right) , \\ \sinh kz &= \frac{1}{2}\left(e^{kz} - e^{-kz}\right) ,\end{aligned}$$

both of which can be approximated as $\frac{1}{2}e^{kz}$ (except close to the bottom) as $kD \to \infty$. This means that the solutions decay downwards from the surface as $\exp(-k(D - z))$, and becoming vanishingly small at depth, as shown in Fig. 2.5. Thus, though the waves can propagate great distances horizontally, they remain **trapped** near the surface and do not penetrate deep into the water. So, the water depth becomes irrelevant. This is because all the

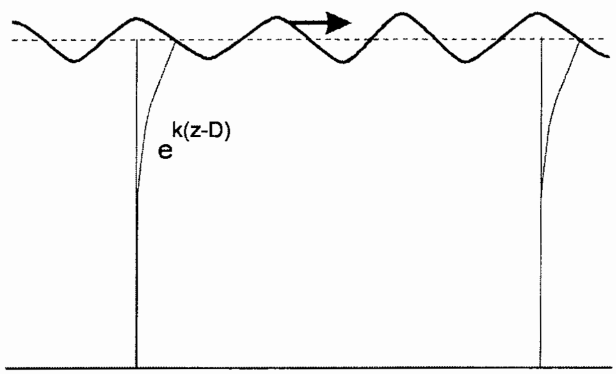

Figure 2.5: Surface waves on deep water.

dynamics of the waves—the effective elasticity that allows their existence—is tied up with the density discontinuity at the water surface. Thus, the surface acts as a **wave guide**, channeling the wave propagation.

2.3 Background theory—dispersive waves

2.3.1 Dispersion

The general dispersion relation for the frequency ω of 1-D waves of wavenumber k can be written

$$\omega = \omega(k) \tag{2.22}$$

and the phase speed is

$$c = \frac{\omega(k)}{k} \ . \tag{2.23}$$

Clearly, c is independent of k for all k only if $\omega(k)$ =constant$\times k$—this is the nondispersive case we discussed earlier and, as we saw, it implies that all disturbances, including localized ones, propagate without change of shape. This can be thought of in terms of Fourier components. Any non-sinusoidal disturbance can be described a sum of components of different wavenumber; if all these waves propagate at the same speed, so will the disturbance itself, and its shape will not change.

For many kinds of wave motion, however (including surface waves on deep water), c varies with k, in which case the different wavenumber components will have different speeds, a phenomenon known as **wave dispersion**.

Therefore the way they interfere with one another will change with time—so the shape of the disturbance will change.

2.3.2 Group velocity

There is one particularly important aspect of dispersion, which concerns the way that modulations propagate on a wave train.

A monochromatic wave of the form

$$\chi(x,t) = Re\left[A\ e^{i[k_0 x - \omega(k_0)t]}\right]$$

with single wavenumber k_0 has the simple spectrum of a δ-function $A\delta(k-k_0)$ [Fig. 2.6(a)], and behaves in the way we have discussed, propagating with a speed $c = \omega(k_0)/k_0$.

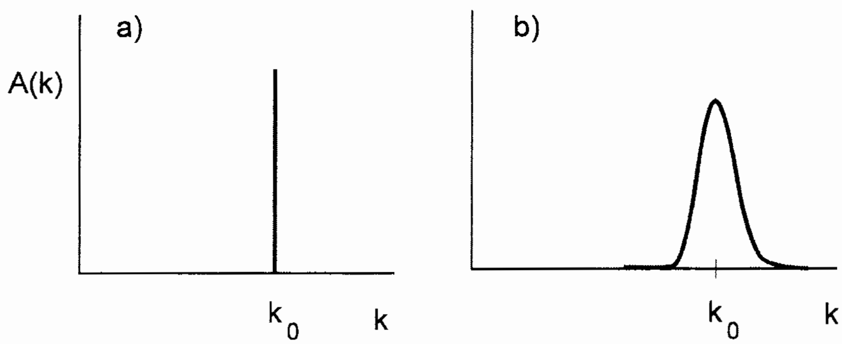

Figure 2.6: Wavenumber spectra for (a) a monochromatic wave, and (b) an almost-monochromatic wave.

Consider now an almost monochromatic wave, with a narrow spectrum [Fig. 2.6(b)]. This can be written

$$\chi(x,t) = Re \int_0^\infty A(k)\ e^{i[kx-\omega(k)t]}\ dk\ , \tag{2.24}$$

where $A(k)$ is nonzero only for wavenumbers in the vicinity of k_0. Writing $k = k_0 + \delta k$, we can rewrite this as

$$\chi(x,t) = Re \int_0^\infty A(k_0+\delta k)\ e^{i[k_0 x-\omega(k_0)t]}\ e^{i[\delta k\ x - \delta\omega\ t]} dk\ ,$$

where $\delta\omega = \omega(k_0 + \delta k) - \omega(k_0) \simeq (\partial\omega/\partial k)\delta k$. At $t = 0$, this is simply

$$\chi(x,t) = F(x)\; e^{ik_0 x}$$

where

$$F(x) = Re\int_0^\infty A(k_0 + \delta k)\; e^{i\,\delta k\,x}\; dk$$

is the modulating envelope of the wave train, which has carrier wavenumber k_0, as illustrated in Fig. 2.7.

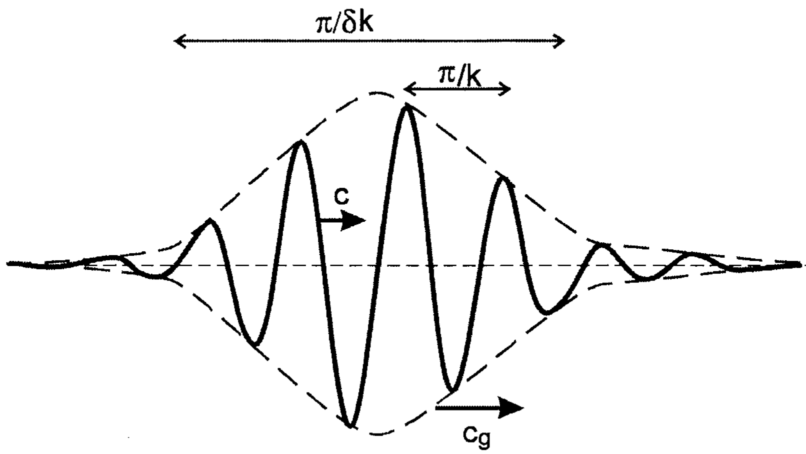

Figure 2.7: An almost-monochromatic wave packet, comprising many wavelengths. The phase of the carrier wave propagates at the phase speed, but the modulation envelope propagates at the group velocity.

Now, for $t > 0$, the wave packet behaves as

$$\begin{aligned}\chi(x,t) &= \left\{Re\int_0^\infty A(k_0+\delta k)\; e^{i[\delta k(x-c_g t)]} dk\right\}\; e^{i[k_0(x-ct)]} \\ &= F(x-c_g t)\; e^{i[k_0(x-ct)]}\,, \qquad (2.25)\end{aligned}$$

where

$$c_g = \left.\frac{\partial\omega}{\partial k}\right|_{k=k_0} \qquad (2.26)$$

is the **group velocity**. Thus, while the carrier wave propagates at the phase speed, the modulation envelope propagates at the group velocity. This is an important concept, as it is the latter velocity that governs the propagation of information, as we shall see.

Nondispersive waves have $\omega = c_0 k$, with constant phase speed c_0, and so their group velocity is the same as their phase velocity. But the group velocity of dispersive waves differs from the phase speed, so in a wave packet like that shown in Fig. 2.7 the wave crests will move at a different speed than the envelope. If $c > c_g$ (which, as we shall see, is the case for deep water waves), new wave crests appear at the rear of the wave packet, move forward through the packet, and disappear at its leading edge. We shall see some examples of this below.

In general, it is easy to get a feel for both phase and group propagation graphically from the dispersion relation, as shown in Fig. 2.8.

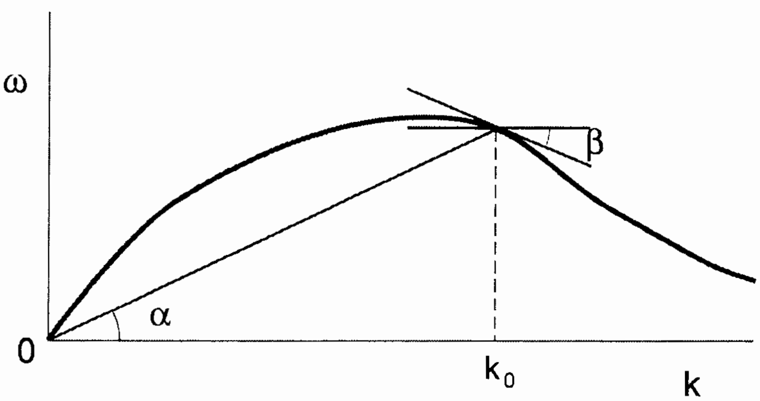

Figure 2.8: The disperion relation $\omega(k)$. The phase velocity at $k = k_0$ is $\tan\alpha$, the group velocity $\tan\beta$.

For any wavenumber k, the phase velocity is just given by the slope α of a line joining the point (ω, k) to the origin, while the group velocity is given by the slope $\tan\beta$ of the tangent to the curve at (ω, k).

2.3.3 Group velocity in multi-dimensional waves

For future reference, we note here that for 2- or 3-dimensional waves, the general dispersion relation is of the form

$$\omega = \omega(\mathbf{k}) \tag{2.27}$$

where $\mathbf{k} = (k, l, m)$ is the (2-D or 3-D) vector wavenumber. The phase velocity[4] is

$$\mathbf{c} = \left(\frac{\omega}{k}, \frac{\omega}{l}, \frac{\omega}{m}\right) , \tag{2.28}$$

and the group velocity

$$\mathbf{c}_g = \left(\frac{\partial\omega}{\partial k}, \frac{\partial\omega}{\partial l}, \frac{\partial\omega}{\partial m}\right) . \tag{2.29}$$

As we shall see later (and a comparison of (2.28) and (2.29) implies), not only may $\mathbf{c}_g$ and $\mathbf{c}$ be of different magnitudes, they may also be *in different directions.*

2.4 Surface wave dispersion

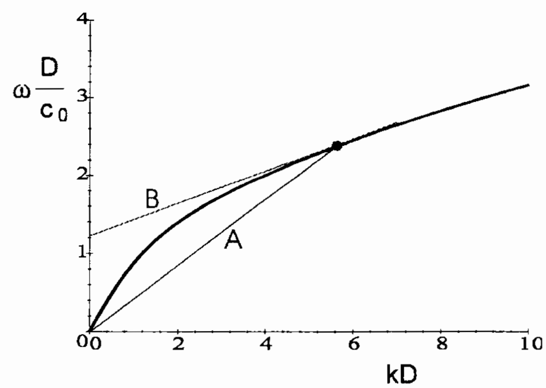

Figure 2.9: Phase speed c(slope of line A)and group velocity c_g (slope of line B)for surface waves.

Returning now to surface waves, and the dispersion relation (2.19) shown in Fig. 2.4, we can see from Fig. 2.9 that $c_g \leq c$ for all wavenumbers (the slope of line B never being greater than that of line A. This is shown more explicitly in Fig. 2.10.

[4]The phase velocity is in fact not a vector, even though it has magnitude and direction. It does not transform like a vector under rotation—this stems from the fact that phase propagation has no meaning along the phase lines.

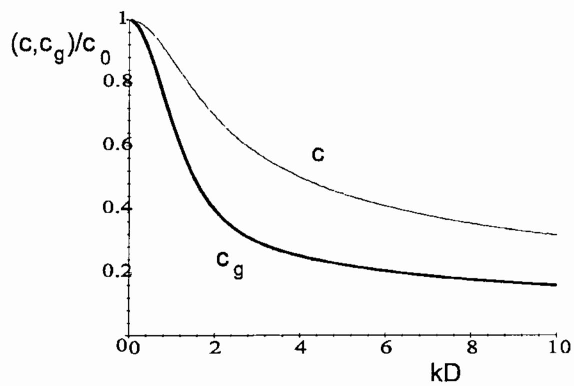

Figure 2.10: Scaled phase and group velocities for surface waves.

In fact, both group and phase speeds are greatest and equal in the long wave (shallow water) limit, when

$$c,\ c_g \to \sqrt{gD}\ , \qquad kD \to 0\ . \tag{2.30}$$

Note that in this limit c becomes independent of k, *i.e.*, the waves become nondispersive, as we saw in the shallow water case. However, as Figs. 2.9 and 2.10 make clear, c and c_g differ significantly for $kD \geq 1$. In the short wave (deep water) limit, in fact, from (2.21),

$$c \simeq 2c_g \to \sqrt{\frac{g}{k}}\ , \quad kD \to \infty\ . \tag{2.31}$$

The difference between nondispersive long waves and dispersive short waves is illustrated in the following. Consider an initial disturbance to the water surface of the form

$$h'(x,0) = \exp\left(-\left(\frac{x}{\Delta D}\right)^2\right)\ .$$

If Δ is large, this has a length scale long compared with D (so it projects primarily onto waves with $kD < 1$) and, as shown in Fig. 2.11 for $\Delta = 4$, the waves emanating from the disturbance are essentially nondispersive.

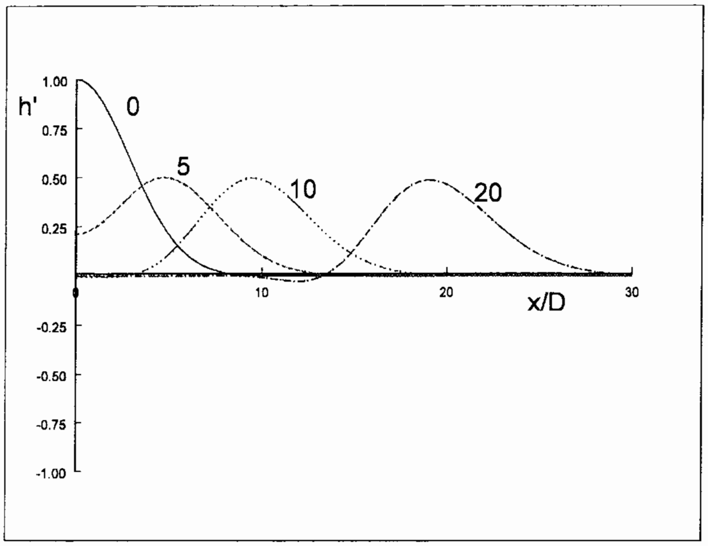

Figure 2.11: Dispersion of a large-scale initial disturbance [exp $\left(-\left(x/4D\right)^2\right)$on water of depth D. Behavior is symmetric about $x = 0$; only $x > 0$ is shown. Numbers on curves are time in units of D/c_0.

(There is just a hint of dispersion; note the negative tail at $t = 20D/c_0$. Note also that there is, for $t > 0$, an identical disturbance, not shown, in $x < 0$.) When Δ is smaller however, the initial disturbance has a smaller length scale and hence a greater projection onto the dispersive short waves. This is illustrated in Fig. 2.12, for which the initial disturbance has $\Delta = 0.5$.

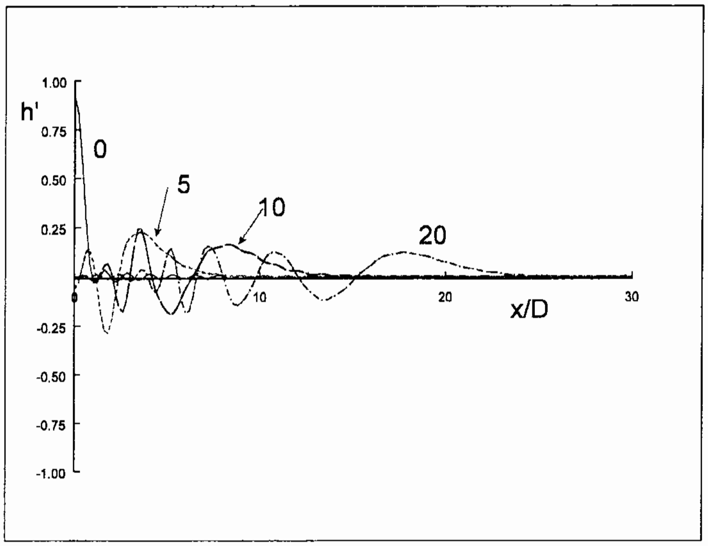

Figure 2.12: Same as the previous figure , but for a small-scale initial disturbance $[\exp(-(2x/D)^2)]$.

The dispersion of the resulting waves is evident. Note especially that (*e.g.*, at $t = 20D/c_0$) the leading edge of the wave train has the largest wavenumber, consistent with our observation that the long waves have the greatest group velocity and, in fact, should travel a distance $x = 20D$ by this time). Wavelength becomes progressively shorter in the tail of the wave train.

Both examples can easily be reproduced in a container of water (one that is large enough to allow the dispersion to develop) or outdoors in a river or lake. If the water is sufficiently shallow, in response to a localized disturbance (*e.g.*, from a small object dropped into the water) you will see a localized wave propagating away, with little or no apparent dispersion. If the water is deep, however, you will see a dispersing wave train, with the longest waves at the leading edge. If you look closely, you may be able to see the wave crests appearing in the rear and propagating forward through the wave train (since $c > c_g$)[5]. This effect can also be seen in a ship's wake (Fig. 2.13):

[5]Though in this case they never overtake the front of the wave train, since this consists of long waves for which $c = c_g$.

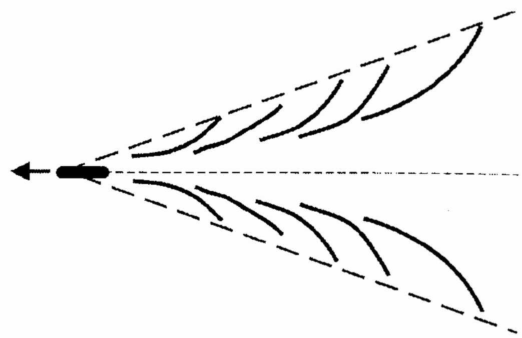

Figure 2.13: Waves in the wake of a ship.

individual wave crests can be seen propagating to the outside of the wake (changing wavenumber in the process), and disappearing there. (Incidentally, in deep water the half-angle of the "wedge" made by the wake is $\arcsin(\frac{1}{3}) = 19.5°$, independent of the speed of the ship. This can be shown to follow from the fact that $c_g/c = \frac{1}{2}$ for deep water; we will not pursue that here but the proof can be found in many texts, such as "Waves in Fluids", Lighthill, Cambridge U P, 1978; section 3.10.)

The results that $c_g \leq c$ is made apparent in another common phenomenon, the wave train downstream of an obstacle in flowing water (*e.g.*, a river), as shown in Fig. 2.14.

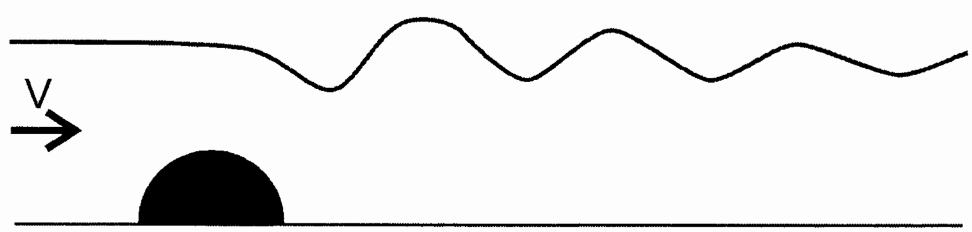

Figure 2.14: Disturbance produced by flow over an obstacle.

There are two key statements to be made:

(i) the wave train is **stationary**, and so has $c = 0$;

(ii) it is located *downstream*—there is no disturbance ahead of the obstacle.

Since our analysis has been for waves on stationary water, let's shift our frame of reference to move with the flow. So now the obstacle and its wave train are moving to the left with speed V. Since the wave crests are stationary with respect to the obstacle, they are also moving to the left with phase speed, in this frame of reference, $c' = -V$. Now, the energy is radiated from the moving obstacle radiates away at the group velocity, *which is never greater than* $|c'| = V$—so it must lag behind the moving obstacle. To put it another way, in this moving frame of reference, $\left|c'_g\right| \leq V$ (remember that c_g is negative in this case). So, in the frame in which the obstacle is stationary, the group velocity is $c_g = c'_g + V \geq 0$. Thus, wave energy can only radiate downstream and there are no upstream effects[6].

There is another aspect to this problem that is illustrative of the general characteristics of waves in fluids. The obstacle is subject to a **force** associated with the flow across it. For a small, smooth obstacle, this force is not primarily frictional (though there is a component of that) but is mostly the result of **form drag**: the pressure on the upstream side of the obstacle is greater than that on the downstream side, and so the obstacle is subjected to a force to the right. (On Fig. 2.14, the free surface height is greater upstream of the peak of the obstacle than at a comparable position downstream; therefore there is a positive pressure gradient on the obstacle.) Simultaneously, of course, the water must be subjected to an equal force to the left (decelerating the flow). However, this is felt, *not* at the obstacle but downstream, within the wave train: this can happen because (like, *e.g.*, electromagnetic waves) the waves, which as we have seen are capable of transporting energy, can also effect **transport of momentum**. Thus, the drag on the obstacle is relayed to remote parts of the water. This has several ramifications for atmospheric and oceanic dynamics.

[6]The fact that this result is not trivial is underlined by the observation that it does not always apply. Capillary waves—those for which surface tension is crucial—*can* have $c_g > c$ and so can and do travel upstream. If the obstacle is small ($<$ a few cm in size) these waves are important though, as they have small wavelength, they dissipate quickly and so may be hard to see.

2.5 Particle motions within a wave

Consider now the motion of (neutrally buoyant) marked particles in the water, which is of course the same thing as the motion of the water itself. Let the instantaneous position of a particle be $(x, z) = (X, Z) + (\eta', \zeta')$ where (X, Z) is the undisturbed position (where the particle would be in the absence of the wave), and η' and ζ' are the small perturbations in position associated with the wave motion. From the definition of velocity as rate of change of position, we have

$$\begin{aligned} \frac{dx}{dt} &= u \, , \\ \frac{dz}{dt} &= w \, ; \end{aligned}$$

or (since (X, Z) is fixed in time)

$$\begin{aligned} \frac{\partial \eta'}{\partial t} + u' \frac{\partial \eta'}{\partial x} + w' \frac{\partial \eta'}{dz} &= u' \, , \\ \frac{\partial \zeta'}{\partial t} + u' \frac{\partial \zeta'}{\partial x} + w' \frac{\partial \zeta'}{dz} &= w' \, . \end{aligned}$$

Linearizing, we have

$$\begin{aligned} \frac{\partial \eta'}{\partial t} &= u' \, ; \\ \frac{\partial \zeta'}{\partial t} &= w' \, . \end{aligned}$$

Now, suppose we have a single propagating wave (single wavenumber, single frequency). Then, from (2.13), (2.14), and (2.18), we have

$$\begin{pmatrix} u' \\ w' \end{pmatrix} = Re \left[-\frac{k}{\rho} Q_0 \begin{pmatrix} i \, \cosh kz \\ \sinh kz \end{pmatrix} e^{i(kx - \omega t)} \right] . \qquad (2.32)$$

Therefore

$$\begin{pmatrix} \eta' \\ \zeta' \end{pmatrix} = Re \left[\frac{k}{\omega \rho} Q_0 \begin{pmatrix} \cosh kz \\ i \, \sinh kz \end{pmatrix} e^{i(kx - \omega t)} \right] . \qquad (2.33)$$

Note:

1. The displacements are oscillatory, so there is no net drift of the particles. Thus, even though the wave pattern propagates, fluid parcels do not: they merely oscillate about their mean position.

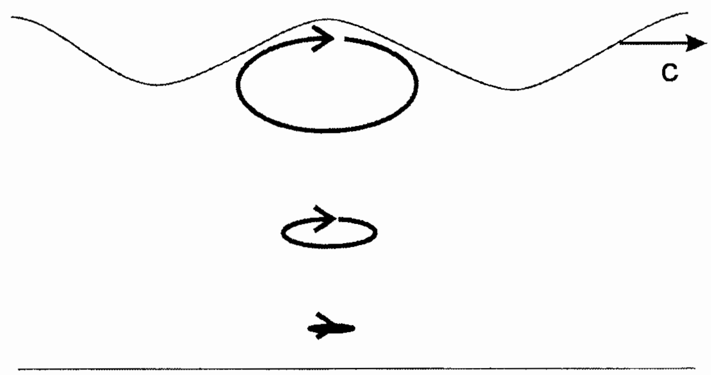

Figure 2.15: Parcel orbits in a water wave. Direction of motion is reversed for a wave traveling to the left.

2. Vertical and horizontal displacements are *in quadrature* ($\pi/2$ out of phase). The marked particles perform elliptical orbits, with, for real $Q_0 > 0$ (which we can insist on, by defining the time origin accordingly) and $k > 0$,

$$\begin{pmatrix} \eta' \\ \zeta' \end{pmatrix} = -\frac{kQ_0}{\omega\rho} \times \begin{bmatrix} \cosh kz \ \cos(kx - \omega t) \\ \sinh kz \ \sin(kx - \omega t) \end{bmatrix} ,$$

which implies that the parcels move clockwise around the orbit for $\omega > 0$ ($c > 0$) , and anticlockwise for $\omega < 0$ ($c < 0$). See Fig. 2.15. Note that the ratio of vertical to horizontal axes of the ellipse is $\tanh kz$, which increases from zero at the bottom (where the boundary ensures that $\zeta' \to 0$) to $\tanh kD < 1$ at the top. For waves in deep water ($kD \gg 1$), the orbits become circular.

2.6 Wave generation by wind

It is common experience that waves on the ocean and lakes are usually weak on calm days, but strong on windy days, suggesting that much of the surface wave activity is somehow produced by the action of wind. It seems unlikely, though, that a wave of wavenumber k and frequency ω is *directly* forced by winds as this would require that the wind itself (or the pressure fluctuations that accompany the wind) have a significant component at the same frequency and wavenumber. There will always be some such component—especially in the turbulent atmospheric boundary layer, the winds have a rich spectrum—but, at the short wavelengths and high frequencies characteristic of surface water waves, these fluctuations are generally weak. Moreover, laboratory studies show that waves can be generated by blowing a *steady* air flow across a water surface. (Try blowing across a glass of water.)

The underlying process common to most wave generation is one of **instability**. That is, even though there may be no externally imposed "waviness" in the wind, there is a tendency to amplify any small perturbation on the water surface. Consider Fig. 2.16.

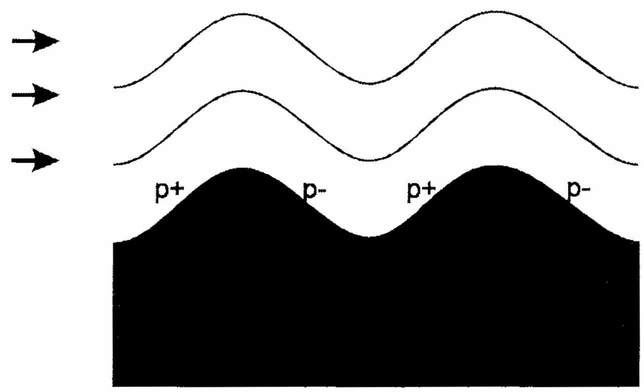

Figure 2.16: Wave generation by wind (schematic). A wave on the ocean surface disturbs the air flow in such a way a to produce a pressure perturbation at the surface that reinforces the wave.

In the absence of a surface wave, the air flow is uniform and the pressure on the water surface is uniform; there is thus no tendency to force waves in the water. However, in the presence of a small wave on the surface, the air flow is disturbed and, like the "rock-in-the-river" problem[7], a perturbed pressure gradient is produced at the water surface which has the same wavenumber (and frequency) as the surface perturbation. Provided the water-air system can get the phase relationships right (and it can) these perturbations reinforce will each other and grow, thus producing, eventually, a finite-amplitude surface wave from an initially infinitessimal perturbation.

[7]Except now the "rock" is the bump on the water surface and the "river" is the atmosphere.

2.7 Wave breaking

All our discussion thus far has been based on our linear (small-amplitude) approximation to the full problem. Using this approximation, we have been able to explain many of the commonly observed properties of surface water waves. There is, however, at least one familiar aspect of these waves that cannot be explained by linear theory: breaking, which is most commonly observed as waves run up a beach.

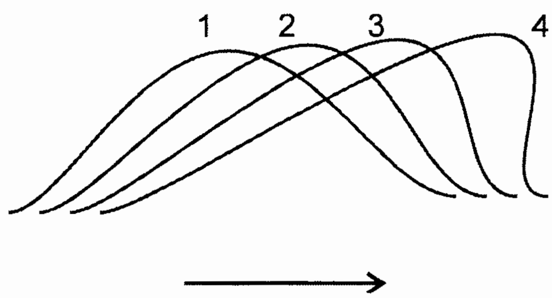

Figure 2.17: Steepening of a finite amplitude wave.

This happens for two reasons. First, as the waves run up the beach, the energy in the wave becomes focused into a shallower layer (once $D \leq k^{-1}$ or so), thus concentrating the energy and increasing the wave amplitude. Second, when $D \leq k^{-1}$ or so, the phase speed becomes dependent on depth, being greater where the water is deeper. In a finite amplitude wave, the water is deeper at the wave crest than in the wave trough. Hence the crests travel faster than the troughs; the crest therefore tend to catch up with and, eventually, overtake the troughs, as shown in Fig. 2.17. This produces the overturning of the wave that is familiar in breakers.

2.8 Further reading

See the suggestions given in Section 1.6.