12.510 Introduction to Seismology
Spring 2008

For information about citing these materials or our Terms of Use, visit: http://ocw.mit.edu/terms.
Feb. 29th 2008:

We have introduced the equations:

\[ \nabla^2 \phi(x, t) = \frac{1}{\alpha^2} \frac{\partial^2 \phi(x, t)}{\partial t^2} \quad \text{and} \quad \nabla^2 \psi(x, t) = \frac{1}{\beta^2} \frac{\partial^2 \psi(x, t)}{\partial t^2} \]

With \( \alpha = [(\lambda + 2\mu)/\rho]^{1/2} \) and \( \beta = (\mu/\rho)^{1/2} \)

These equations can be solved using 3 different methods:

1. D’Alembert’s solution (most ‘physical’ approach)
2. Separation of variables
3. Fourier Transforms (mathematically, the most powerful method)

There is a whole class of theoretical development in applied maths that uses Fourier Integral Operators (FIOs)

1. D’Alembert’s Solution:

Take as an example, the wave equation:

\[ \ddot{\phi} = \alpha^2 \nabla^2 \phi \]

And the function:

\[ \phi(x, t) = f(x - ct) + g(x + ct) \quad (45) \]

The first term: \( f(x - ct) \) represents propagation in the positive \( x \)-direction

The second term: \( g(x + ct) \) represents propagation in the negative \( x \)-direction

\( c = \) the wave speed or phase speed/velocity

Consider the profile of the wave at a time \( t_0 \) and at some later time \( t_1 \)

Figure 4:
(x-ct) is known as the phase of the wave.

The phase speed is given by: \( c = \frac{x_1 - x_0}{t_1 - t_0} \)  \( (46) \)

**Figure 5: Diagram to illustrate the concept of wavefronts:**

A wavefront is a line in 2d (or surface in 3d) connecting points of equal phase.

In reality, the wavefronts are circular, but locally they behave as a plane wave.

All points along the wave-front have the same travel-time from the origin.

The relationship between the wavenumber \((k)\) and angular frequency \((\omega)\) is given by:

\( k = \frac{\omega}{c} \)  \( (47) \)

The relationship between wavenumber \((k)\) and wavelength \((\lambda)\) is given by:

\( k = \frac{2\pi}{\lambda} \)  \( (48) \)

So we can re-write the phase in terms of wavenumber:

\( \left( \frac{x}{c} - t \right) \rightarrow (kx - \omega t) \) (in one dimension)

In 3 dimensions, this becomes: \( (k_x x + k_y y + k_z z - \omega t) \) or \( (k \cdot x - \omega t) \)

The harmonic function is a solution to the wave equation:
\[ \phi = \cos(kx - \omega t) + i \sin(kx - \omega t) = \exp(i(kx - \omega t)) \quad (49) \]

Where we have used the identity: \( \exp(i\alpha) = \cos(\alpha) + i\sin(\alpha) \) \quad (50)

2. Separation of variables:

Using the method of separation of variables, we trial a solution of the wave equation of the form:

\[ \phi(x, y, z, t) = X(x)Y(y)Z(z)T(t) \quad (51) \]

Substituting this into the equation: \( \ddot{\phi} = c^2 \nabla^2 \phi \) gives:

\[ \frac{1}{X} \frac{d^2 X}{dx^2} + \frac{1}{Y} \frac{d^2 Y}{dy^2} + \frac{1}{Z} \frac{d^2 Z}{dz^2} - \frac{1}{c^2} \frac{d^2 T}{dt^2} = 0 \quad (52) \]

To satisfy this equation, each term must be equal to a constant and the constants must sum to zero. We choose the constants: \(-k_x^2, -k_y^2, -k_z^2, \left(\frac{\omega^2}{c^2}\right)\) respectively

\[ \frac{d^2 X}{dx^2} + k_x^2 X = 0 \rightarrow e^{\pm ik_x x} \]
\[ \frac{d^2 Y}{dy^2} + k_y^2 Y = 0 \rightarrow e^{\pm ik_y y} \]
\[ \frac{d^2 Z}{dz^2} + k_z^2 Z = 0 \rightarrow e^{\pm ik_z z} \]
\[ \frac{d^2 T}{dt^2} + \omega^2 T = 0 \rightarrow e^{\pm i\omega t} \quad (53) \]

Applying the condition that these constants must sum to zero gives us the dispersion relation:

\[ k_x^2 + k_y^2 + k_z^2 - \left(\frac{\omega^2}{c^2}\right)^2 = 0 \quad \text{dispersion relationship} \quad (54) \]

Substituting the solutions for X,Y,Z,T back into the original trial solution:

\[ \phi(x, y, z, t) = X(x)Y(y)Z(z)T(t) \]

Gives our final solution of the form: \( \phi = e^{i(k \cdot x - \omega t)} \quad (55) \)

Note: Wavenumbers and the wavevector:
We have already defined:

\[ k = \frac{2\pi}{\lambda} \]

The wavevector is: \( k = (k_x, k_y, k_z) \) and gives the “direction” of the wave.

The length of the wavevector is the wavenumber: \( |k| = \frac{\omega}{c} = k \) \((56)\)

Our solution is effectively harmonic functions that propagate in the direction of \( k \).

The full solution is a superposition of plane waves:

The displacement is related

\[ \phi(k, x) = \sum_{k, \omega} e^{i(k \cdot x - \omega t)} \] \((57)\)

\[ u = \nabla \phi \] \((58)\)

\[ u(x, t) = \nabla \phi(x, t) \] \((59)\)

So \( u(x, t) = (0, 0, ik_z)A \exp\{i(k_z z - \omega t)\} \) \((60)\)

The displacement vector has harmonic wave character and propagates in the z direction.

The imaginary part of the displacement vector is associated with the amplitude of the wave.

The propagating part is found by taking the real part of this displacement vector.

\[ u(x, t) = (0, 0, ik_z)A \exp\{i(k_z z - \omega t)\} = ik_x [\cos(k_z z - \omega t) + i \sin(k_z z - \omega t)] \] \((61)\)

The real part of this is: \( \text{Re} [u(x, t)] = -k_z \sin(k_z z - \omega t) \) \((62)\)

---

**Note: The Helmholtz equation:**

If we consider solutions to the wave equation: \( \ddot{\phi} = \alpha^2 \nabla^2 \phi \) in 1d, of the form \( \phi = e^{i(k \cdot x - \omega t)} \)

We can differentiate with respect to time get:

\[ \frac{d^2 \phi}{dt^2} = -\omega^2 \phi \] \((63)\) Substituting into the wave equation gives:

\[ \nabla^2 \phi + \frac{\partial^2}{\partial x^2} \phi = 0 \] \((64)\) and \( k = \frac{\omega}{\alpha} \) so, we have:

\[ \nabla^2 \phi + k^2 \phi = 0 \] (The Helmholtz equation) \((65)\)
A lot of imaging is done in the frequency domain by finding solutions to the helmholtz equation.

3. Fourier Transforms:

Fourier transforms allow us to understand the relationship between the space-time \((x,t)\) and wavenumber-frequency \((k,\omega)\) domains.

In one dimension, the forward and reverse fourier transforms between the space-frequency and space-time domains are given by:

\[
\Phi(x, \omega) = \int_{-\infty}^{\infty} \phi(x, t) e^{i \omega t} dt \longleftrightarrow \phi(x, t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(x, \omega) e^{-i \omega t} d\omega \quad (66)
\]

(Note: in seismology, we normally take the exponential as having a positive sign when we are transforming into the space-frequency domain, however this is simply a convention)

Similarly, we can use fourier transforms to convert between the space-time and wavenumber-time domains. In 3d the fourier transforms between the space-time and wavenumber-time domains are:

\[
\Phi(k, t) = \int \phi(x, t) e^{i k \cdot r} d^3 r \longleftrightarrow \phi(x, t) = \frac{1}{(2\pi)^3} \int \Phi(k, t) e^{-i k \cdot x} dk_x dk_y dk_z \quad (67)
\]

Combining these gives the double-fourier transform:

\[
\phi(x, t) = \frac{1}{(2\pi)^3} \int_{-\pi}^{\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(k_x, k_y, k_z, \omega, z) e^{i(k \cdot x - \omega t)} dk_x dk_y dk_z \quad (68)
\]

Note that \(k_z\) does not appear in this equation. This is because, \(k_x, k_y, k_z\) are related via the dispersion relation.

\[
k_x^2 + k_y^2 + k_z^2 - \left(\frac{\omega}{c}\right)^2 = 0 \text{ dispersion relationship.} \quad (69)
\]

Hence, if we have specified the angular frequency, \(k_x\) and \(k_y\), it follows that \(k_z\) has already been determined.

Numerically, this double fourier transform is very difficult to work with.
A synthetic seismogram is given by a plane-wave superposition.

Suppose we want to make a synthetic seismogram that looks similar to the wave.

We do not need to integrate over the full range \(-\pi < k_x < \pi\) and \(-\pi < k_y < \pi\), because this implies we do not know anything about the direction of the wave.

A synthetic seismogram can be produced by limiting the integration over directions \(k_0 \pm dk\) and frequency \(\omega_0 \pm d\omega\).

\[
\phi(x, t) = \frac{1}{(2\pi)^3} \int_{\omega_0-d\omega}^{\omega_0+d\omega} \int_{k_0-dk}^{k_0+dk} \Phi(k_x, k_y, \omega, z) e^{i(k \cdot x - \omega t)} dk_x dk_y d\omega
\]

The integrand \(\phi(k_x, k_y, \omega z)\) is the amplitude or weight.

**Slowness:**

We have see already that the modulus of the wave-vector gives the wavenumber:

\[
|k| = \sqrt{k_x^2 + k_y^2 + k_z^2} = \frac{W}{c}
\]

In 2d, we have:

\[
|k| = \sqrt{k_x^2 + k_z^2} = \frac{W}{c}
\]  

(71)

**Figure 6:**

In figure 6, the arrow is used for a ray and the dashed line is used for a wavefront. The wavenumber \(k\) indicates the direction of the ray. The angle \(i\) is both the ‘take-off’ angle and the ‘angle of incidence’

For a P-wave, the wavenumber is given by: \(k_\alpha = \frac{\omega}{\alpha}\) and for the S-wave, the wavenumber is given by \(k_\beta = \frac{\omega}{\beta}\)

Since, in general: \(\beta < \alpha\) it follows directly, that: \(k_\beta = \frac{\omega}{\beta} > k_\alpha = \frac{W}{\alpha}\)  

(72)
Since \( k_a = \frac{\omega}{\alpha} \) gives the length of the vector representing the P-wave and \( k_p = \frac{\omega}{\beta} \) represents the length of the vector representing the S-wave, it follows directly that P-waves 'dive' less steeply into the medium than S-waves.

**Figure 7:**

![Diagram](diagram.png)

The phase 'speed' \( c \), is given by: \( c = \frac{ds}{dt} \) (73) and is a vector in the direction of propagation.

At the surface, we measure: \( c_x = \frac{dx}{dt} \) (74) which is the 'apparent' velocity/speed.

**Horizontal slowness:**

\[
\sin(t) = \frac{ds}{dx} = c \frac{dt}{dx} = c \left( \frac{1}{c_x} \right) = cp \quad (75)
\]

\[
\rho = \frac{1}{c_x} = \frac{\sin(t)}{c} = \text{horizontal slowness} = \text{ray parameter} \quad (76)
\]

This follows from Snell’s law.

**Vertical slowness:**

We know that: \( c_x = (c_x, c_z) \)

The vertical slowness is given by: \( \eta = \frac{1}{c_z} = \frac{\cos(i)}{c} \) \quad (77)

Combining the vertical and horizontal slowness:

\[
\rho^2 + \eta^2 = \frac{\sin^2 t}{c^2} + \frac{\cos^2 t}{c^2} = 1 \quad (78)
\]

Rearranging this gives:

\[
\eta = \sqrt{\frac{1}{c^2} - \rho^2} \quad (79)
\]
So, the vertical slowness does change with depth, because \( c \) is a function of depth. The vertical slowness (\( \eta \)) is zero if \( \frac{1}{c^2} = p^2 \) (which represents a horizontally propagating wave).

\( \eta \) is imaginary for evanescent waves. (This is important for understanding the behaviour of surface waves).

There is a direct relationship between the wave-vector and the slowness components:

\[
    k = \frac{\omega}{c}
\]

\[
    k_x = \frac{\omega}{c_x} = \omega p \quad (80)
\]

\[
    k_z = \frac{\omega}{c_z} = \omega \eta \quad (81)
\]

\[
    k = (k_x, k_z) = (\omega p, \omega \eta) = \omega (p, \eta) \quad (82)
\]

Notes: Katie Atkinson, Feb 2008

Figures from notes of Patricia M Gregg (Feb 2005) Kang Hyeun Ji (Feb 2005)