Elasticity

So far:

Stress → angle of repose vs accretionary wedge

Strain → reaction to stress → but how?

Constitutive relations

\[ \tau_{ij} = \tau_{ij}(\varepsilon_{kl}) ; \quad \varepsilon_{ij} = \varepsilon_{ij}(\tau_{kl}) \]

For example,

Elasticity

- Isotropic
- Anisotropic

Viscous flow

- Isotropic
- Anisotropic

Power law creep

Viscoelasticity

Trade offs:

\begin{align*}
\text{simplicity} & \leftrightarrow \text{realism} \\
\text{constant} & \leftrightarrow \text{variable} \\
\text{isotropic} & \leftrightarrow \text{anisotropic} \\
\text{elastic, viscous} & \leftrightarrow \text{viscoelastic} \\
\text{history} & \leftrightarrow \text{history dependent} \\
\text{independent} &
\end{align*}
Tensors

Most physical quantities that are important in continuum mechanics like temperature, force, and stress can be represented by a tensor. Temperature can be specified by stating a single numerical value called a scalar and is called a zeroth-order tensor. A force, however, must be specified by stating both a magnitude and direction. It is an example of a first-order tensor. Specifying a stress is even more complicated and requires stating a magnitude and two directions—the direction of a force vector and the direction of the normal vector to the plane on which the force acts. Stresses are represented by second-order tensors.

Tensors are quantities independent of coordinate system.
\[ \alpha_{ij} = \cos \phi_{ij} \]

where \( \phi_{ij} \) is the angle of primed to original.

\[ x_i' = \alpha_{ij} x_j \]
\[ x_i = \alpha_{ji} x_j' \]
\[ \alpha_{ij} = \frac{\partial x_i'}{\partial x_j} = \frac{\partial x_j}{\partial x_i'} \]

Tensors:

a. 0\(^{\text{th}}\) order (scalar) – quantity dependent only on position
b. 1\(^{\text{st}}\) order (3\(^1\) components) \( A_i' = \alpha_{ij} A_j \)
c. 2\(^{\text{nd}}\) order (3\(^2\) = 9 components) \( A_{ij}' = \alpha_{ik} \alpha_{jk} A_{sk} \)
d. 3\(^{\text{rd}}\) order (3\(^3\) = 27 components) \( A_{ijk}' = \alpha_{is} \alpha_{jr} \alpha_{kp} A_{sp} \)
e. 4\(^{\text{th}}\) order (3\(^4\) = 81 components) \( A_{ijkl}' = \alpha_{is} \alpha_{jr} \alpha_{kp} \alpha_{iq} A_{spq} \)

Conventional moduli:

1. Hydrostatic comp.
\[ \tau_{ij} = -p \delta_{ij} \]
\[ \tau_{ii} = -p \delta_{ii} = 3\lambda e_{kk} + 2\mu e_{ii} \]
\[ = -3p = (3\lambda + 2\mu)e_{ii} \]
\[ -\frac{p}{e_{ii}} = -\frac{VP}{\Delta V} \equiv K \]

where \( K = \lambda + 2/3\mu \) is bulk modulus.

2. Uniaxial stress

\[ \tau_{11} = T \]
other \( \tau_{ij} = 0 \)
\[ 2\mu e_1 = T - \frac{\lambda}{2\mu + 3\lambda} T \]
\[ \frac{T}{e_1} \equiv E \] (sometimes \( Y \))

where \( E = \frac{\mu(2\mu + 3\lambda)}{\mu + \lambda} \) is Young’s modulus

Hook’s law:

\[ T = E e \]
\[ \frac{e_{22}}{e_{11}} = \frac{e_{33}}{e_{11}} \equiv -\nu \] This is called Poisson’s ratio.
\[ 2\mu e_{22} = -\frac{\lambda}{2\mu + 3\lambda} \tau_{11} \quad \Rightarrow \quad \nu = \frac{\lambda}{2(\mu + \lambda)} \]
\[ \theta = e_{11} + e_{22} + e_{33} = e_{11}(1 - 2\nu) \]
fluid: \( \mu \to 0 \quad \Rightarrow \quad \nu \to \frac{1}{2} \)

most material: \( \nu = 0.2 - 0.3 \)
\[ \nu = \frac{1}{4} \quad \Rightarrow \quad \lambda = \mu \] It is Poisson solid.
steel: \( \nu \); 0.3 – 0.33

seismically measured

\[

v_p = \sqrt{\frac{\lambda + 2\mu}{\rho}} \quad v_s = \sqrt{\frac{\mu}{\rho}}

\]

compare \( v_p, v_s \rightarrow \nu \rightarrow \) discriminate rock types

3. Simple shear

\[

\tau_{12} = \tau_{21} = \tau \\
\tau_{12} = 2\mu e_{12} = 2Ge_{12}

\]

where \( G \) is shear modulus.

Note: Among \( \lambda, \mu, K, \nu, E, G \) only two are independent.