Navier-Stokes Equation – dimensional form

\[ \frac{\partial p}{\partial x_i} + \eta \frac{\partial^2 v_i}{\partial x_j \partial x_j} + f_i = \rho \frac{Dv_i}{Dt} \]

Assume:

- Characteristic velocity \( v_0 \)
- Characteristic length \( L \)
- Characteristic stress \( \eta v_0 / L \)
- Characteristic time \( v_0 / L \)

Choose non-dimensional variables

\[ v' = v / v_0 \quad x' = x / L \quad p' = \frac{L}{\eta v_0} p \quad t' = \frac{v_0}{L} t \]

or

\[ v = v' v_0 \quad x = x' L \quad p = \frac{\eta v_0}{L} p' \quad t' = \frac{L}{v_0} t' \]

\[ \frac{\partial}{\partial x} = \frac{1}{L} \frac{\partial}{\partial x'} \quad f' = \frac{L^2}{\eta v_0} f \quad \frac{\partial}{\partial t} = \frac{v_0}{L} \frac{\partial}{\partial t'} \]

Substitute into Navier-Stokes equation

\[ \frac{1}{L} \frac{\eta v_0}{L} \frac{\partial p'}{\partial x'_i} + \frac{1}{L^2} v_0 \eta \frac{\partial^2 v'_i}{\partial x'_j \partial x'_j} + \frac{1}{L^2} v_0 \eta f'_i = \rho v_0 \frac{v_0}{L} \frac{Dv'_i}{Dt'} \]

or

\[ \frac{\partial p'}{\partial x'_i} + \frac{\partial^2 v'_i}{\partial x'_j \partial x'_j} + \frac{\rho v_0 L}{\eta} \frac{Dv'_i}{Dt'} \]
where \( \frac{\rho v_0 L}{\eta} = \text{Re} \)

⇒ Re gives importance of inertial terms relative to viscous terms.

\( \text{Re} \ll 1 \quad \text{viscous forces} \approx \text{balance} \)

acceleration negligible

\( \text{Re} \gg 1 \quad \text{inertia dominates} \)

Note: in dimensionless form, Re is the only parameter in the Navier-Stokes equation.

⇒ for given geometry (boundary conditions) ALL equivalent

(non-dimensional) problems at same Re give same result!

Examples:

1. Low Reynolds number flow past a cylinder.

\( \text{Re} \ll 1 \quad \text{Symmetry, like in the sphere problem.} \)
2. Re = 10 \( \frac{\partial v_i}{\partial t} = 0 \) (steady) \( v_j \frac{\partial v_i}{\partial x_j} \neq 0 \)

Asymmetry; eddies in wade.

Figure 23.2
Figure by MIT OCW.

The figure below is experimental.

Figure 23.3
Figure by MIT OCW.

Figure 23.3. Variation of drag coefficient with Reynolds number for circular cylinder.
Re ≤ 1 \[ \rightarrow \quad C_D \sim 1 / Re \quad D \sim V \]

\[ 10^2 \leq Re \leq 3 \cdot 10^5 \]
\[ \rightarrow \quad C_D \sim \text{const.} \quad D \sim V^2 \]

Re \sim 3 \cdot 10^5 \[ \rightarrow \quad \text{big drop in } C_D! \]

Recall

Earth’s mantle: Re \sim 10^{-19}

canoe: Re \sim 2 \cdot 10^5

Summary:

For low Re, inertia not important

Navier-Stokes equation linear

“simple” results (analytic theory)

For high Re, inertia important

Navier-Stokes equation nonlinear

time dependent

complicated – experimental approach \rightarrow empirical