Why aren’t subducted slabs curled over or vertical?

What is the flow pattern beneath oceanic ridges?

Does constant thickness basaltic crust make sense?

Corner flow models:

- Batchelor – Introduction to fluid mechanics
- McKenzie – GJRAS 18, 1-32, 1969
- Tovish et al, JGR 83, 5892, 1978
- McAdoo, JGR, 87, 8684, 1982

Slab model:

\[ 0 = \eta \nabla^2 \psi + \rho \nabla U - \nabla p \]

\[ 0 = \nabla \cdot \psi \]

Use vorticity

\[ \omega = \nabla \times \psi \]

Take curl of Stokes equation (heading toward 4th order equation)

\[ 0 = \eta \nabla \times \nabla^2 \psi + \nabla \times (\rho \nabla U - \nabla p) \]

Recall \( \nabla \times (\nabla \phi) = 0 \), \( \nabla \cdot (\nabla \times A) = 0 \)

Assume \( \rho \) constant

Recall \( \nabla^2 \psi = \nabla (\nabla \cdot \psi) - \nabla \times \nabla \times \psi \)
\[ 0 = -\eta (\nabla \times \nabla \times \nabla \times \nu) \]
\[ = -\eta (\nabla \times \nabla \times \omega) \]
\[ = \eta \nabla^2 \omega \]

Next – define stream function \( \psi \) by \( \nu = \nabla \times \psi \)

Automatically satisfies \( \nabla \cdot \nu = 0 \)

Then \( \omega = \nabla \times \nabla \times \psi = -\nabla^2 \psi \)

For our purposes, consider 2-D flow only

\[ \psi = (0, 0, \psi) \quad r, \theta, z \]

Must solve scalar equation

\[ \nabla^2 (\nabla^2 \psi) = 0 \]

\[ \nabla^4 \psi = 0 \quad \text{← Biharmonic equation [Recall Airy stress function]} \]

To help in guessing solution, recall

\[ v_r = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \]
\[ v_\theta = - \frac{\partial \psi}{\partial r} \]

\[ \tau_{rr} = -p + 2\eta \frac{\partial v_r}{\partial r} \]
\[ \tau_{\theta\theta} = -p + 2\eta \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) \]
\[ \tau_{r\theta} = \eta \left( \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{\partial v_\theta}{\partial r} - \frac{v_\theta}{r} \right) \]

\[ \nabla^2 \psi = \frac{1}{r} \frac{\partial \psi}{\partial r} \left( r \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \psi}{\partial \theta^2} + \frac{\partial^2 \psi}{\partial z^2} \]

Let’s try solutions

\[ \psi = R(r) \Theta(\theta) \]

We will want solutions with \( v_r \) independent of \( r \)

Try \( R(r) = r \)

\[ \psi = r \Theta \]

Plugging in \( \Rightarrow \frac{d^4 \Theta}{d\theta^4} + 2\frac{d^2 \Theta}{d\theta^2} + \Theta = 0, \)

or \( \Theta'''' + 2\Theta'' + \Theta = 0 \)

Solution:
$$\Theta = A \sin \theta + B \cos \theta + C \sin \theta + D \theta \cos \theta$$

Now – match boundary conditions
\[ v_r = \Theta' = A \cos \theta - B \sin \theta + C (\sin \theta + \theta \cos \theta) + D (\cos \theta - \theta \sin \theta) \]
\[ v_\theta = -\Theta \]

Slab problem –
Break into 2 regions

<table>
<thead>
<tr>
<th>Back-arc region</th>
<th>Fore-arc region</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\theta = 0$</td>
<td>$\theta = 0$</td>
</tr>
<tr>
<td>$v_\theta = 0 = B$</td>
<td>$v_\theta = 0$</td>
</tr>
<tr>
<td>$v_r = 0 = A + D$</td>
<td>$v_r = -v$</td>
</tr>
<tr>
<td>$\theta = \theta_b$</td>
<td>$\theta = \theta_f$</td>
</tr>
<tr>
<td>$v_\theta = 0 = A \sin \theta_b + C \theta_b \sin \theta_b + D \theta_b \cos \theta_b$</td>
<td>$v_\theta = 0$</td>
</tr>
<tr>
<td>$v_r = v$</td>
<td>$v_r = v$</td>
</tr>
<tr>
<td>$= A \cos \theta_b + C (\sin \theta_b + \theta_b \cos \theta_b) + D (\cos \theta_b - \theta_b \sin \theta_b)$</td>
<td>$= -\Theta$</td>
</tr>
</tbody>
</table>

Need to solve 3 equations, 3 unknowns

**Solutions**

Back-arc
\[ \psi = \frac{rv[(\theta - \theta) \sin \theta \sin \theta - \theta \theta \sin(\theta - \theta)]}{\theta^2 - \sin^2 \theta} \]

Fore-arc
\[ \psi = \frac{-rv[(\theta - \theta) \sin \theta + \theta \sin(\theta - \theta)]}{\theta + \sin \theta} \]

**Stresses**

\[ \frac{\partial v_r}{\partial r} = 0, \quad \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} = 0 \Rightarrow \tau = \begin{pmatrix} -p & \tau_{r\theta} \\ \tau_{r\theta} & -p \end{pmatrix} \]

Back-arc
\[ \tau_{r\theta} = \frac{2v \eta \left[ \theta \cos(\theta - \theta) - \sin \theta \cos \theta \right]}{\theta^2 - \sin^2 \theta} \]

Fore-arc
\[ \tau_{r\theta} = \frac{2v \eta \cos \theta + \cos(\theta - \theta)}{\theta + \sin \theta} \]

Figure 26.3 Stream lines for flow within the mantle. Motion is with respect to the plate behind the island arc and is driven by the motion of the other plate and of the sinking slab. Thermal convection outside the slab is neglected, and the lithosphere is 50 km thick.

Figure by MIT OCW.
**Pressure** – no direct equation

But – equilibrium \( \tau_{ij,j} = 0 \)

\[
- \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} = 0 \\
- \frac{1}{r} \frac{\partial p}{\partial \theta} + 2 \tau_{r\theta} + \frac{\partial \tau_{r\theta}}{\partial r} = 0
\]

Back-arc

\[
p = \frac{2\nu\eta}{r} \frac{\theta_b \sin(\theta_b - \theta) + \sin \theta_b \sin \theta}{\theta_b^2 - \sin^2 \theta_b} < 0
\]

\[
\rightarrow - \frac{2\nu\eta}{r} \frac{\theta_b \sin \theta_b}{\theta_b^2 - \sin^2 \theta_b} \text{ at surface} \quad \rightarrow - \frac{2\nu\eta}{r} \frac{\sin^2 \theta_b}{\theta_b^2 - \sin^2 \theta_b} \text{ on slab}
\]

Fore-arc

\[
p = \frac{2\nu\eta}{r} \frac{\sin \theta - \sin (\theta - \theta_f)}{\theta_f + \sin \theta_f} > 0
\]

\[
\rightarrow \frac{2\nu\eta}{r} \frac{\sin \theta_f}{\theta_f + \sin \theta_f} \text{ at surface and on slab}
\]

\( \Rightarrow \) excess pressure in fore-arc
suction in back-arc – tendency to lift slab (and also modify surface topography)
This flow tends to lift slab –
Resisted by
1) gravity
2) resistance to squeezing material out of wedge [not well posed for \( \infty \) wedge \( \rightarrow \) needs \( \infty \) work]

Consider gravity – torque balance
\[ T_{\text{flow}} = \int_0^l (P_f - P_b) \cdot r \, dr \]
\[ = 2\eta \nu l \left[ \frac{\sin \theta_b}{(\pi - \theta_b) + \sin \theta_b} + \frac{\sin^2 \theta_b}{\theta_b^2 - \sin^2 \theta_b} \right] \]
\[ T_{\text{grav}} = \int_0^l \Delta \rho g h r \cos \theta \, dr \]
\[ = \frac{1}{2} \Delta \rho g h l^2 \cos \theta_b \]

Ridge problem

By symmetry, at \( \theta = \frac{\pi}{2} \), \( v_\theta = 0 \), \( \tau_{r\theta} = 0 \)

Let \( \psi = r\Theta \)

\[ v_r = \Theta' \quad \tau_{r\theta} = \frac{\eta}{r} (\Theta'' + \Theta) \]
\[ v_\theta = -\Theta \quad p = -\frac{\eta}{r} (\Theta'' + \Theta') \]
\[ \nabla^4 \psi = 0 \]
\[ \Theta = A \sin \theta + B \cos \theta + C \sin \theta + D \theta \cos \theta \]

<table>
<thead>
<tr>
<th>( \theta = 0 )</th>
<th>( \theta = \pi/2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v_\theta = 0 \Rightarrow B = 0 )</td>
<td>( v_\theta = 0 \Rightarrow A + (\pi/2)C = 0 )</td>
</tr>
<tr>
<td>( v_r = v = A + D )</td>
<td>( \tau_{r\theta} = 0 \Rightarrow D = 0 )</td>
</tr>
<tr>
<td>( A = v ), ( C = -(2/\pi) v )</td>
<td></td>
</tr>
</tbody>
</table>

\[ v_r = \frac{2v}{\pi} \left[ \frac{\pi}{2} - \theta \right] \cos \theta - \sin \theta \]
\[ \tau_{r\theta} = \frac{4\eta v}{\pi r} \cos \theta \]
\[ v_\theta = \frac{-2v}{\pi} \left[ \frac{\pi}{2} - \theta \right] \sin \theta \]
\[ p = \frac{4\eta v \sin \theta}{\pi r} \]

Figure by MIT OCW.
Note: at $\theta = \frac{\pi}{2}$, $v_r = -\frac{2}{\pi} v$ (related to gradient in stream function)

$\Rightarrow$ upwelling zone wider than horizontal flow zone

$\Rightarrow$ zone of magma generation wider than depth from which melt segregates