Stress

Traction \( T \) on material may depend on area

Body forces \( \mathbf{g} \) depend on volume, neglect in this discussion

- Forces inside body which are a reaction to traction are \textit{stresses}

- Homogeneous if forces acting on a surface of fixed shape and orientation do not depend on position in body.

Then traction on each face of unit cube can be characterized.

For body to be in \textit{static equilibrium}, Traction on (+2 face) = (-2 face) and opposite in fixed otherwise net imbalance causes acceleration governed by Newton's law.

Define stress on each face as traction \textit{/area}

\[ T = \sigma \text{ A} \] i.e. stress tensor relates area normal \( \Rightarrow \) traction

\[ T_i = \sigma_{ij} A_j \]

\[ \sigma_{ij} \triangleq T_i \text{ on } j \text{ face} \]

normal components, \( \sigma_{ii} \) + if traction shear components \( \sigma_{ij}, \epsilon_{ij} \) + if in positive dir on \( i \text{ face} \) + if in negative dir on \( j \text{ face} \)
Eqn. Equilibrium

On x₁ face in x₁ direction

Left:

\[
\sigma_{11} \frac{\partial x_2}{\partial x_1} + \frac{\partial \sigma_{11}}{\partial x_1} \frac{\partial x_2}{\partial x_1} \frac{\partial x_2}{\partial x_3}
\]

\[
 \left[ \sigma_{12} \frac{\partial x_2}{\partial x_1} - \frac{\partial \sigma_{11}}{\partial x_1} \frac{\partial x_2}{\partial x_1} \frac{\partial x_2}{\partial x_3} \right]
\]

result:

\[
\frac{\partial \sigma_{11}}{\partial x_1} (\partial x_1 \partial x_2 \partial x_3)
\]

On x₂ in x₁ direction

Top:

\[
\sigma_{12} (\partial x_1 \partial x_3) + \frac{\partial \sigma_{12}}{\partial x_2} (\partial x_1 \partial x_3)
\]

Bottom:

\[
- \sigma_{12} (\partial x_1 \partial x_3) - \frac{\partial \sigma_{12}}{\partial x_2} (\partial x_1 \partial x_3)
\]

By Newton's law, sum of forces in x direction equals mass/acceleration in that direction:

\[
\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + \epsilon_{x_1} \epsilon_{x_1} = \epsilon_{x_1}^2
\]

In general:

\[
\frac{\partial \sigma_{ij}}{\partial x_j} + \epsilon_{x_i} = \epsilon_{x_i}^2
\]

If \( x_i = 0 \), i.e., in static eqn:

\[
\frac{\partial \sigma_{ij}}{\partial x_i} + \epsilon_{x_i} = 0
\]
Shear moments and symmetry

\[ \int \left( \sigma_{23} + \frac{D\sigma_{23}}{Dx_3} \frac{1}{2} \delta x_3 \right) dx_2 \; dx_3 \]

moment arm for 2 face is \( \frac{1}{2} \delta x_3 \)
moments are anti-clockwise

moment arm for 3 face is \( \frac{1}{2} \delta x_3 \)
moments are clockwise

\[ 2\sigma_{32} \frac{dx_3}{2} (dx_2 dx_1) - 2\sigma_{23} \frac{dx_2}{2} dx_3 dx_1 + \sigma_1 dx_1 dx_2 dx_3 = I \frac{\partial \theta}{\partial \theta} \]

but assume no body torques \( \theta = 0 \)
and note \( I \) order of mag \( \delta x_5 \rightarrow 0 \) as \( dx_1 \)

then \( (\sigma_{32} - \sigma_{23}) dx_1 \; dx_2 \; dx_3 = 0 \)

\[ \Rightarrow \sigma_{32} = \sigma_{23} \]
or \( \sigma_{ij} = \sigma_{ji} \)

\[ \Rightarrow \text{stress tensor is symmetric} \]
Proof that stress is a tensor

Canchy tetrahedron

Total force on face with area \( \text{ABC} \)

\[
\mathbf{T} = \mathbf{p} \cdot (\text{ABC})
\]

In \( x_1 \) direction

\[
p_{1} (\text{ABC}) = \sigma_{11} l_1 + \sigma_{12} l_2 + \sigma_{13} l_3
\]

where \( l_1 = \text{ABC} = \angle x_1 \mathbf{e} \)

\[
sim. \quad p_{2} = \sigma_{21} l_1 + \sigma_{22} l_2 + \sigma_{23} l_3
\]

\[
p_{3} = \sigma_{31} l_1 + \sigma_{32} l_2 + \sigma_{33} l_3
\]

or \( p_{i} = \sigma_{ij} l_j \)

i.e. tensor transformation rule

\( \Rightarrow \) Stress is a 2nd rank, symmetric tensor

\( \Rightarrow \) Principal stresses, \( \text{Quadric} \)

\[
\begin{array}{c}
\sigma_{11} \sigma_{12} \sigma_{13} \\
\sigma_{21} \sigma_{22} \sigma_{23} \\
\sigma_{31} \sigma_{32} \sigma_{33}
\end{array}
\]

\[
\begin{array}{c}
0 \quad 0 \quad 0 \\
0 \quad \sigma_{2} \quad 0 \\
0 \quad 0 \quad \sigma_{3}
\end{array}
\]
i.) uniaxial stress
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

ii.) biaxial stress
\[
\begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

iii.) triaxial stress
\[
\begin{bmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3
\end{bmatrix}
\]

iv.) hydrostatic stress (pressure)
\[
\begin{bmatrix}
-p & 0 & 0 \\
0 & -p & 0 \\
0 & 0 & -p
\end{bmatrix}
\]

v.) Pure shear
\[
\begin{bmatrix}
-\tau & 0 & 0 \\
0 & \tau & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

vi.) Simple shear
\[
\begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]
Summary: Stress Tensor

1.) Traction on a plane with direction cosine \( \zeta \)

\[
T = \sigma \zeta \Rightarrow T_i = \sigma_{ij} \zeta^j
\]

2.) Eqns of motion

\[
\frac{\partial \sigma_{ij}}{\partial x_i} + \rho \ddot{u}_j = \mathbf{m}_j
\]

3.) Stress is symmetric

\( \sigma_{ij} = \sigma_{ji} \)

4.) Principal stress directions

values

\[
\begin{pmatrix}
\sigma_1 & 0 & 0 \\
0 & \sigma_2 & 0 \\
0 & 0 & \sigma_3
\end{pmatrix}
\]

5.) Stress quadratic construction
Stress, Strain, Elasticity:

Nye, chap 5 & 6, pp. 82-105.

1. One dimensional strain
   Relative displacements important
   \[ x \quad P \quad \Delta x \quad Q \]
   \[ x \quad P' \Delta x + \Delta w \quad Q' \]
   inhomogeneous

   \[ \Delta u \]
   homogeneous

Strain at a point \( P \) is

\[ \varepsilon = \frac{\text{increase in length}}{\text{original length}} = \frac{P'Q' - PQ}{PQ} = \frac{\Delta u}{\Delta x} \]

(slope of curve above)

Goal: For a given deformation of body, define a tensor that describes the change of direction and length of any vector in body.

Want to map vector in body (undefined) into vector in body (defined)