Rotating Shallow-Water Waves

We now consider the effects of rotation and boundaries on fluids obeying (in the horizontal) the linearized shallow-water equations

\[
\begin{align*}
\frac{\partial}{\partial t}u + f\mathbf{z} \times u &= -\nabla \phi \\
\frac{\partial}{\partial t} \phi + gH \nabla \cdot u &= 0
\end{align*}
\]

where \( u \) and \( \nabla \) are horizontal vectors/operators.

**Plane waves**

For the simplest case, we take all fields proportional to \( \exp(ik \cdot x - \omega t) \) to find

\[
\begin{pmatrix} -\omega & -f & i\ell \\ f & -i\omega & i\ell \\ gH i\ell & gH\ell & -\omega \end{pmatrix} \begin{pmatrix} u \\ v \\ \phi \end{pmatrix} = 0
\]

which implies

\[
\omega \left[ \omega^2 - f^2 - gH (k^2 + \ell^2) \right] = 0
\]

which has three roots, \( \omega = 0 \) and

\[
\omega^2 = f^2 + gH|\mathbf{k}|^2
\]

which is the generalization of the long gravity wave dispersion relation. In the presence of rotation, the waves become dispersive, with

\[
c_g = \sqrt{gH} \frac{k}{\sqrt{f^2 / gH + |k|^2}}
\]

These can be simplified by using the deformation radius \( \sqrt{gH}/f \) as a length scale

\[
\frac{\omega}{f} = \sqrt{1 + |kR_d|^2}, \quad \frac{c_g}{\sqrt{gH}} = \frac{kR_d}{\sqrt{1 + |kR_d|^2}}
\]

Note that the shallow water equations will only be applicable for

\[
kH = kR_d \frac{H}{R_d} << 1 \quad \Rightarrow \quad kR_d << \frac{\sqrt{g/H}}{f} \sim 500
\]

for a 4000 m deep ocean.
Rotating shallow water waves

Dispersion relation for rotating plane waves $k^* R_d$

The $\omega = 0$ root is non-trivial; to see this, let us look at the equations in vorticity/divergence form. If $\zeta = \mathbf{z} \cdot (\nabla \times \mathbf{u}) = \frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}$ and $D = \cdot \nabla \mathbf{u}$, then

$$\frac{\partial}{\partial t} \zeta + fD = 0$$
$$\frac{\partial}{\partial t} D - f\zeta = -\nabla^2 \phi$$
$$\frac{\partial}{\partial t} \phi + gHD = 0$$

Eliminating $D$ from the first and third equation gives

$$\frac{\partial}{\partial t} \left( \zeta - \frac{f \phi}{gH} \right) = \frac{\partial}{\partial t} q = 0$$
The (linearized) potential vorticity $q = (\zeta - f\frac{\phi}{gH})$ is conserved. This equation implies either the frequency is zero and the potential vorticity is not, or vice-versa. The zero-frequency waves correspond to $D = 0$, $\zeta + \nabla^2 \phi / f$ and are geostrophically balanced. When $f$ varies, these turn into Rossby waves.

If we recast the divergence equation in terms of $q$

$$\frac{\partial}{\partial t} D - f q - \frac{f^2}{gH} \phi = -\nabla^2 \phi$$

and use the conservation of mass equation, we find

$$\frac{\partial^2}{\partial t^2} \phi + f g H q + f^2 \phi = g H \nabla^2 \phi$$

(1)

For the gravity waves with no PV signal,

$$\frac{\partial^2}{\partial t^2} \phi + f^2 \phi = g H \nabla^2 \phi$$

(2)

and we recover the dispersion relation above.

**Adjustment**

If we consider the initial value problem, we can specify the three fields, or, alternatively, we can specify $q(x)$, $\phi(x,0) = \phi_0(x)$, and $\frac{\partial}{\partial t} \phi(x,0) = \phi_{t0}(x)$. Since $q$ remains unchanged, we can split the pressure up into the geostrophic part and the gravity wave part

$$\phi = \phi_g(x) + \phi_w(x, t)$$

$$\nabla^2 \phi_g - \frac{f^2}{gH} \phi_g = \frac{1}{f} q$$

$$\frac{\partial^2}{\partial t^2} \phi_w + f^2 \phi_w = g H \nabla^2 \phi_w$$

$$\phi_w(x,0) = \phi_0(x) - \phi_g(x), \quad \frac{\partial}{\partial t} \phi_w(x,0) = \phi_{t0}(x)$$

The equation for the geostrophic pressure shows that the influence of a localized potential vorticity anomaly spreads out over a scale $R_d = \sqrt{gH/f}$ called the “deformation radius.” I.e., if $q = q_0 \delta(x)$ (independent of $y$), the geostrophic pressure is

$$\phi_g = -\frac{q_0 R_d}{2f} \exp(-|x/R_d|)$$
Gravity waves in a channel

Now we consider waves in a channel $0 < y < W$. In that case, we must apply the boundary conditions $v = 0$ at $y = 0, W$. For the non-rotating case, the $y$-momentum equation implies $\frac{\partial}{\partial y} \phi = 0$ at the boundaries (or, more generally, $\nabla \phi \cdot \hat{n} = 0$). The solutions to the $f = 0$ version of (2) are

$$\phi = \cos(\ell y) e^{i(kx-\omega t)}$$

with

$$\omega^2 = gH(k^2 + \ell^2)$$

and

$$\ell = 0, \frac{\pi}{W}, \frac{2\pi}{W}, \frac{3\pi}{W}, \ldots$$
Frequencies $\omega W/\sqrt{gH}$ for $\ell W = 0, 1, 2, 3, 4, 5$ kW

The rotating case is more complex. If we stick with equation (2), we can use the two momentum equations with $v = 0$ to show that

$$\frac{\partial}{\partial t} u = -\frac{\partial}{\partial x} \phi, \quad f u = -\frac{\partial}{\partial y} \phi \quad \Rightarrow \quad \frac{\partial^2 \phi}{\partial t \partial y} = f \frac{\partial \phi}{\partial x}$$

For waves with $\phi = \Phi(y) \exp(ikx - \omega t)$, we must satisfy

$$\frac{\partial}{\partial y} \Phi = -\frac{f k}{\omega} \Phi \quad \text{at} \quad y = 0, \ W$$

and

$$gH \frac{\partial^2}{\partial y^2} \Phi = (f^2 + gHk^2 - \omega^2) \Phi$$

We can look for solutions $\Phi = \cos(\ell y + \theta)$; the dispersion relation is then the same as for plane waves, but the boundary conditions imply

$$\ell \sin \theta = \frac{f k}{\omega} \cos \theta \quad \text{and} \quad \ell \sin(\ell W + \theta) = \frac{f k}{\omega} \cos(\ell W + \theta) \quad \Rightarrow \quad \tan(\theta) = \tan(\ell W + \theta)$$
Thus $\ell W$ is still an integer multiple of $\pi$. However, the $\ell = 0$ solution is no longer satisfactory, since it makes $\Phi$ constant, which will not be consistent with the boundary conditions.

The dispersion relation

$$\omega^2 = f^2 + gH (k^2 + \ell^2) = f^2 + gH \left( k^2 + \frac{n^2 \pi^2}{W^2} \right)$$

now correspond to modes with the same cross channel wavelength as before, but which no longer have their maxima at the channel walls:

$$\tan \theta = \frac{fk}{\ell \omega}$$
A sample waveform for $kW = \pi$, $\ell W = \pi$, $f/\sqrt{(gH)} = 5$ is

![Sample waveform](image)

**Kelvin waves**

But we can also look for exponential solutions; we can see that

$$
\Phi = \exp \left(-\frac{f k}{\omega} y \right)
$$

clearly satisfies the boundary conditions. Starting with the general case $A e^{\alpha y} + B e^{-\alpha y}$ leads to the conclusion that the solution above is the only correct one. Putting this into the equation for $\Phi$ gives

$$
g H \frac{f^2 k^2}{\omega^2} = f^2 + g H k^2 - \omega^2
$$

which has the solutions

$$
\omega^2 = g H k^2 , \quad \omega^2 = f^2
$$

The latter is spurious; if we examine the momentum equations with $\omega = f$, we find $u = \frac{f}{k} \phi$ since the other solution $v = i \frac{k}{f} \phi$ will not satisfy the boundary conditions. The mass equation then implies $f^2 = g H k^2$ which is not generally correct. Thus, we find that the $\ell = 0$ mode is replaced by one which decays across the channel as

$$
\Phi = \exp \left(-\frac{f}{\sqrt{gH}} y \right) = \exp \left(-\frac{y}{R_d} \right)
$$
and has frequency

$$\omega = \sqrt{gHk}$$

These non-dispersive waves are called Kelvin waves.