Shallow-water or long waves

For surface gravity waves, we can simplify the equations for the case of long waves (or shallow-water waves) from either the potential or the original momentum equations.

Potential

Our basic nonlinear equations in the case where the bottom depth varies $H = H_0 + h(x,t)$ become

$$\nabla^2 \phi = 0$$

$$\frac{\partial h}{\partial t} - \nabla \phi \cdot \nabla h = \phi_z \quad \text{at} \quad z = -H_0 - h(x,t)$$

$$\frac{\partial \eta}{\partial t} - \nabla \phi \cdot \nabla \eta = -\phi_z \quad \text{at} \quad z = \eta(x,t)$$

$$\frac{\partial \phi}{\partial t} = g\eta + \frac{1}{2} |\nabla \phi|^2 \quad \text{at} \quad z = \eta$$

If we nondimensionalize $z$ by $H_0$, $x, y$ by $L$, $\eta$ by $\eta_0$, $t$ by $L/\sqrt{gH_0}$, $h$ by $h_0$ and $\phi$ by $g\eta_0 L/\sqrt{gH_0}$, we get

$$\frac{\partial^2 \phi}{\partial z^2} + \delta^2 \nabla^2 \phi = 0$$

$$\epsilon_h \delta^2 \frac{\partial h}{\partial t} - \epsilon_h \delta^2 \nabla \phi \cdot \nabla h = \epsilon \phi_z \quad \text{at} \quad z = -1 + \epsilon_h h(x,t)$$

$$\delta^2 \frac{\partial \eta}{\partial t} - \delta^2 \epsilon \nabla \phi \cdot \nabla \eta = -\phi_z \quad \text{at} \quad z = \epsilon \eta(x,t)$$

$$\frac{\partial \phi}{\partial t} = \eta + \epsilon \frac{1}{2} \left( \frac{\partial \phi}{\partial z} \right)^2 + \epsilon |\nabla_h \phi|^2 \quad \text{at} \quad z = \epsilon \eta$$

with $\delta = H_0/L$, $\epsilon = \eta_0/H_0$, and $\epsilon_h = h_0/H_0$. For the long-wave limit, we take $\delta^2 \ll 1$ and $\epsilon, \epsilon_h \sim 1$ (at least by comparison). Then the lowest order equations tell us that

$$\frac{\partial^2 \phi_0}{\partial z^2} = 0 \quad , \quad \frac{\partial \phi}{\partial z} = 0 \quad \text{at} \quad z = -1 + \epsilon_h h \quad , \quad \epsilon \eta$$

for which the solution is $\phi_0 = \Phi(x, y, t)$. This is consistent with the dynamic equation also. At the next order ($\delta^2$), we find

$$\frac{\partial^2 \phi_1}{\partial z^2} = -\nabla^2_h \Phi$$

$$\epsilon_h \frac{\partial h}{\partial t} - \epsilon_h \epsilon \nabla \phi \cdot \nabla h = \epsilon \frac{\partial}{\partial z} \phi_1 \quad \text{at} \quad z = -1 + \epsilon_h h(x,t)$$

$$\frac{\partial \eta}{\partial t} - \epsilon \nabla \phi \cdot \nabla \eta = -\frac{\partial}{\partial z} \phi_1 \quad \text{at} \quad z = \epsilon \eta(x,t)$$

$$\frac{\partial \Phi}{\partial t} = \eta + \epsilon |\nabla_h \Phi|^2 \quad \text{at} \quad z = \epsilon \eta$$

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Integrating Poisson’s equation in $z$ and applying the boundary conditions gives the mass conservation equation

$$\left( \frac{\partial}{\partial t} - \epsilon \nabla \Phi \right) \vec{H} = \epsilon \vec{H} \nabla^2 \Phi$$

with the nondimensional depth of the fluid being $\bar{H} = 1 + \epsilon \eta + \epsilon \eta \epsilon$. The dynamic equation is

$$\frac{\partial}{\partial t} \Phi = \eta + \epsilon |\nabla \Phi|\^2$$

If we look at linear, flat-bottom waves $h = 0$, $\epsilon << 1$ (but now requiring $\delta^2 << \epsilon << 1$), we have

$$\frac{\partial}{\partial t} \eta = \nabla^2 \Phi$$

$$\frac{\partial}{\partial t} \Phi = \eta$$

giving the nondimensional wave equation

$$\frac{\partial^2}{\partial t^2} \eta = \nabla^2 \eta$$

**From basic equations**

For rotating stratified flow, we have

$$\frac{\partial}{\partial t} \mathbf{u} + (\zeta + f \hat{z}) \times \mathbf{u} = -\nabla (P + \frac{1}{2} |\mathbf{u}|^2) + b \hat{z}$$

$$\nabla \cdot \mathbf{u}_h + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial}{\partial t} b + \mathbf{u} \cdot \nabla b = 0$$

From the momentum equations, we can form a vorticity equation (Cartesian form)

$$\frac{\partial}{\partial t} Z_i + \nabla_j (u_j Z_i) - \nabla_j (u_i Z_j) = \nabla \times b \hat{z} = -\hat{z} \times \nabla b$$

or

$$\frac{\partial}{\partial t} Z_i + \mathbf{u} \cdot \nabla Z_i - \mathbf{Z} \cdot \nabla u_i = -\hat{z} \times \nabla b$$

with $\mathbf{Z} = \zeta + f \hat{z}$. The flow can be irrotational when $f = 0$ and $b = 0$: a non-rotating, constant density fluid.
Hydrostatic

If \( L \gg H_0 \), then the continuity equation implies \( w \sim \frac{H_0}{L} u_h \) and the \( w \) terms in the \( x \) and \( y \) components of \( \zeta \) are order \( \delta^2 \) compared to the others:

\[
\zeta_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \sim -\frac{\partial v}{\partial z}
\]

so that the vorticity in the momentum equations is replaced by \( \zeta_h = \nabla \times u_h \). Likewise the \( w^2 \) term in the Bernouilli function is order \( \delta^2 \) compared to the others. Finally, if \( P \sim UL/T \) then

\[
\frac{[\frac{\partial}{\partial t} w]}{[\frac{\partial}{\partial z} P]} \sim \frac{U H_0/LT}{UL/H_0T} = \delta^2
\]

Dropping all the \( \delta^2 \) terms gives

\[
\frac{\partial}{\partial t} u_h + (\zeta_h + f\hat{z}) \times u = -\nabla (P + \frac{1}{2}|u_h|^2) + b\hat{z}
\]

Note that vertical advection is still significant:

\[
[w \frac{\partial}{\partial z}] = U \frac{H_0}{L} \frac{1}{H_0} \sim [u_h \frac{\partial}{\partial x}]
\]

In conventional form, we have

\[
\frac{D}{Dt} u_h + f\hat{z} \times u_h = -\nabla_h P \quad , \quad \frac{\partial}{\partial z} P = b
\]

Homogeneous fluid

In this case, if the horizontal vorticities are zero initially, they will remain so; i.e. at time 0

\[
\frac{\partial}{\partial t} \zeta_1 + u \cdot \nabla \zeta_1 - (\zeta_3 + f) \frac{\partial}{\partial z} u_1 = 0 = \frac{\partial}{\partial t} \zeta_1 + u \cdot \nabla \zeta_1 - (\zeta_3 + f) \zeta_2
\]

implying \( \frac{\partial}{\partial t} \zeta_1 = 0 \). Thus we have

\[
\frac{\partial}{\partial z} u_h = 0
\]

The vertical momentum equation implies \( \frac{\partial}{\partial z} P = 0 \) and the continuity equation tells us that \( \frac{\partial}{\partial z} w \) is independent of depth so that

\[
\frac{\partial}{\partial z} w = \frac{w(\eta(x, t)) - w(-H(x, t))}{H(x, t) + \eta(x, t)} = \frac{1}{H + \eta} \left( \frac{\partial}{\partial t} + u_h \cdot \nabla \right) (H + \eta)
\]

Finally, we note that the pressure at the surface is

\[
-\rho \rho g \eta(x, y, t) + \rho_0 P(x, y, t) = p_o(x, y, t)
\]
where \( p_a \) is the atmospheric pressure. Thus

\[
P = g\eta + \frac{1}{\rho_0}p_a
\]

and our equations become

\[
\frac{\partial}{\partial t} \mathbf{u}_h + (\zeta_3 + f)\mathbf{z} \times \mathbf{u}_h + \nabla (\frac{1}{2}|\mathbf{u}_h|^2) = \frac{D}{Dt} \mathbf{u}_h + f\mathbf{z} \times \mathbf{u}_h = -\nabla g\eta - \nabla \frac{p_a}{\rho_0}
\]

\[
\frac{\partial}{\partial t}(H + \eta) + \nabla \cdot [\mathbf{u}_h(H + \eta)] = 0
\]

These are the “shallow water equations”

**Irrotational case**

When \( f = 0 \), \( \zeta_3 \) will also stay zero, and we can use

\[
\mathbf{u}_h = -\nabla \Phi
\]

and the momentum equations give

\[
\frac{\partial}{\partial t} \nabla \Phi = \nabla (g\eta + \frac{p_a}{\rho_0} + \frac{1}{2}|\nabla \Phi|^2) \quad \text{or} \quad \frac{\partial}{\partial t} \Phi = g\eta + \frac{1}{2}|\nabla \Phi|^2 + \frac{p_a}{\rho_0}
\]

and

\[
\frac{\partial}{\partial t}(H + \eta) - \nabla \cdot [(H + \eta)\nabla \Phi] = 0
\]

as before.