Objective analysis [scalar fields]

We would like to estimate a property \( S(x) \) at a point \( x \) given a set of observations \( s(x_i) \) (containing errors) of the property at other spatial points. In the absence of other information (climatology, for example), we assume that the data mean represents the true mean and use a linear estimator for the deviations

\[
\hat{S}'(x) = \sum s'(x_i)a(x_i, x)
\]

or, using summation convention,

\[
\hat{S}'(x) = s'(x_i)a(x_i, x)
\]  

(1)

The problem now becomes the choice of \( a \).

We form an error estimate

\[
\epsilon = \frac{1}{2} \langle [\hat{S}'(x) - S'(x)]^2 \rangle
\]

\[
= \frac{1}{2} \langle s'(x_i)s'(x_j)a(x_i, x)a(x_j, x) - \langle S'(x)s'(x_i) \rangle a(x_i, x) + \frac{1}{2} \langle S'(x)S'(x) \rangle \rangle
\]  

(2)

We seek the minimum error with respect to the values of the coefficients \( a(x_i, x) \)

\[
\frac{\partial \epsilon}{\partial a(x_i, x)} = 0
\]

which implies

\[
\langle s'(x_i)s'(x_j) \rangle a(x_j, x) = \langle S'(x)s'(x_i) \rangle \]

(3)

The symmetry of \( \langle s'(x_i)s'(x_j) \rangle \) has been used. We write this in terms of the covariance for the field

\[
C(x - x') = \langle s'(x)s'(x') \rangle
\]

assuming that the measurement noise is uncorrelated and has variance \( \sigma^2 \)

\[
[C(x_i - x_j) + \sigma^2 \delta_{ij}]a(x_j, x) = C(x_i - x)
\]

If we know or can approximate the covariance function, we can set up and solve this linear system to give \( a(x_i, x) \) for any target point \( x \)

\[
a(x_j, x) = [C(x_i - x_j) + \sigma^2 \delta_{ij}]^{-1}C(x_i - x)
\]  

(4)

Not only can we substitute this in (1) to find the estimated field at \( x \), we can also get an estimate of the error by using (3) and (4) in (2)

\[
\epsilon = \frac{1}{2} C(0) - \frac{1}{2} C(x - x_j)[C(x_i - x_j) + \sigma^2 \delta_{ij}]^{-1}C(x_i - x)
\]

Note that the errors depend only on the sampling positions and \( C, \sigma \). Therefore we can design sampling strategies given an estimate of the covariance and the noise.