Two-D Turbulence – Homework 2

Structure functions/ Covariances

For homogeneous, isotropic turbulence, we can write the covariance in terms of \( f(r) \) which is the longitudinal covariance (the expected value of the product of the components of velocity along the the line joining the two points; i.e., parallel to \( r \)) and \( g(r) \) which is the transverse covariance (the expected value of the product of the velocities normal to \( r \)). If \( r = r\hat{x} \) then

\[
\langle u(x, y)u(x + r, y) \rangle = f(r) , \quad \langle u(x, y)v(x + r, y) \rangle = 0 , \quad \langle v(x, y)v(x + r, y) \rangle = g(r)
\]

Why is the second true?

a) Use these to show in general that

\[
\langle u_i(x)u_j(x + r) \rangle = \frac{f(r) - g(r)}{r^2} r_i r_j + g(r) \delta_{ij}
\]  

(1)

b) Applying the continuity equation to the covariance implies

\[
\frac{\partial}{\partial r_j} \langle u_i(x)u_j(x + r) \rangle = 0
\]

Use this and eqn. (1) to find the relationship

\[
\frac{\partial}{\partial r} (rf) = g
\]

in two dimensions. (What is it in three?)

c) In two dimensions, the flow is given by a streamfunction

\[
u = -\frac{\partial}{\partial y} \psi , \quad v = \frac{\partial}{\partial x} \psi
\]

so that we can relate the transverse and longitudinal covariances to \( C(r) \equiv \langle \psi(x)\psi(x + r) \rangle \).

Use again \( r = r\hat{x} \) to find the relationship between \( g \) and \( C \).

d) From the t256 turbulence simulation, find \( C \) and estimate \( f \) and \( g \) from that. The MATLAB program qgproc2 gives a start on the problem. I've looked at about \( t = 8 \), but you can look at other times as well. The qgprocm.m program will display the fields. The files are available on-line as

http://lake.mit.edu/~glenn/12.822t/t256.in
http://lake.mit.edu/~glenn/12.822t/t256.out
http://lake.mit.edu/~glenn/12.822t/qgprocm.m
http://lake.mit.edu/~glenn/12.822t/qgproc2.m

e) What does \( S_2(r) \) look like and what do you think about the practicalities of computing of \( S_2(r) \)?

f) In your copious free time, you can look at the runs with beta (t256b1 and t256b5) to see what differences there might be and how anisotropy shows up. [optional...]