Convection problem set

1. Hele-Shaw Cell Convection

   The Hele-Shaw cell has a thin layer of fluid confined between two glass or plexiglass plates at \( y = 0 \) and \( y = \delta \). It allows us to study real analogues to 2-D flows.

   1) Assume that \( u \) and \( w \) have the characteristic parabolic profile in \( y \)

   \[
   u(x, y, z, t) = u(x, z, t) f(y) \quad \text{and} \quad w(x, y, z, t) = w(x, z, t) f(y) \quad , \quad f(y) = \frac{y}{\delta} \left( 1 - \frac{y}{\delta} \right)
   \]

   and that pressure and buoyancy are independent of \( y \) and \( v = 0 \). Take the \( y \) average of the scaled equation:

   \[
   \frac{1}{Pr} \frac{D}{Dt} u = -\nabla p + Ra \, b \, \ddot{z} + \nabla^2 u \\
   \nabla \cdot u = 0 \\
   \frac{D}{Dt} b = \omega + \nabla^2 b
   \]

   (\( b \) being the deviation from \( \bar{b} = -z \)). Show that the averaged momentum equation reduces to

   \[
   \frac{12}{\delta^2} u = -\nabla p + Ra \, b \, \ddot{z}
   \]

   when \( \delta \) is very small (thin fluid layer) and the Rayleigh number is big.

   2) Define the streamfunction and the \( y \)-component of the vorticity. Eliminate the pressure and write coupled equations for \( \psi \) and \( b \).

   3) Solve the linear stability problem and find the critical Rayleigh number assuming free-slip conditions on top and bottom,

   4) Consider now the nonlinear problem. We could solve for weakly supercritical conditions by assuming \( \epsilon < Ra - Ra_0 \) is small and expanding in suitable powers. However, it is easier to follow Lorenz and examine a truncated systems. Derive the equivalents to the Lorenz equations for this system. You can use the same expansions for \( b \) and the vorticity as in the notes. Show that the resulting equations have a simple bifurcation from the motionless state to a stable steady state.
2. Line Plum

A line plume is driven by a steady buoyancy flux per unit length $G_0$ (units of $m^3 s^{-3}$) on a line which extends infinitely far in one horizontal direction $y$. The resultant plume is therefore 2-dimensional, with no variation in the $y$-direction. Entrainment into the plume occurs only in the $x$-direction. In this problem, we’ll examine the derivation of the plume equations and the solutions.

Let the turbulent plume extend from $-r(z) < x < \tilde{r}(z)$. If the boundary were an impermeable surface, then we would have

$$u = w \frac{\partial r}{\partial z}$$

but entrainment adds an extra flow which we’ll call $-u_e$ (negative since the entrainment is going inward)

$$u = w \frac{\partial r}{\partial z} - u_e$$

Integrate the two-D continuity equation from 0 to $\tilde{r}$ and show that the averaged vertical velocity satisfies

$$W = \frac{1}{r} \int_0^r dx w$$

2) Now consider the flux of buoyancy, integrating from 0 to $\tilde{r}$ where the outside buoyancy value is $b_e$. Assume you can replace the average of $wb$ with $W$ times $B$, the average of $b$.

3) Finally treat the vertical momentum. The additional assumption is that the pressure is the same as the environment.

4) Now look for similarity solutions: assume $W \sim z^n$, $r \sim z^m$ and find the solutions given a uniform background buoyance $b_e \sim 0$. Make the entrainment approximation $u_e = \alpha W$, where $\alpha$ is a constant.

5) Suppose you can assess the effects of rotation by finding the height $z_f$ beyond which the plume Rossby number $Ro = W/(fr)$ is less than 1 using the similarity solutions above. What is $z_f$ and what do you think might happen at greater heights?