1 Short Run Production Function

In the short run, the capital input is fixed, so we only need to consider the change of labor. Therefore, the production function

\[ q = F(K, L) \]

has only one variable \( L \) (see Figure 1).

Average Product of Labor.

\[ AP_L = \frac{\text{Output}}{\text{Labor Input}} = \frac{q}{L} \]

Slope from the origin to (L,q).

Marginal Product of Labor.

\[ MP_L = \frac{\partial \text{Output}}{\partial \text{Labor Input}} = \frac{\partial q}{\partial L} \]

Additional output produced by an additional unit of labor.

Some properties about \( AP \) and \( MP \) (see Figure 2).

- When \( MP = 0 \),
  \( \text{Output is maximized.} \)
- When \( MP > AP \),
  \( AP \text{ is increasing.} \)
1 Short Run Production Function

Figure 1: Short Run Production Function.

Figure 2: Average Product of Labor and Marginal Product of Labor.
2 Long Run Production Function

- When \( MP < AP \),

  \( AP \) is decreasing.

- When \( MP = AP \),

  \( AP \) is maximized.

To prove this, maximize \( AP \) by first order condition:

\[
\frac{\partial}{\partial L} \frac{q(L)}{L} = 0
\]

\[\Rightarrow \quad \frac{\partial q}{\partial L} \frac{1}{L} - \frac{q}{L^2} = 0\]

\[\Rightarrow \quad \frac{\partial q}{\partial L} = \frac{q}{L}\]

\[\Rightarrow \quad MP = AP.\]

*Example* (Chair Production.). Note that here \( AP_L \) and \( MP_L \) are not continuous, so the condition for maximizing \( AP_L \) we derived above does not apply.

<table>
<thead>
<tr>
<th>Number of Workers</th>
<th>Number of Chairs Produced</th>
<th>( AP_L )</th>
<th>( MP_L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
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<td>N/A</td>
</tr>
<tr>
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<td>2</td>
<td>2</td>
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</tr>
<tr>
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<td>9</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Table 1: Relation between Chair Production and Labor.

2 Long Run Production Function

Two variable inputs in long run (see Figure 3).

*Isoquants.* Curves showing all possible combinations of inputs that yield the same output (see Figure 4).

*Marginal Rate of Technical Substitution* (\( MRTS \)). Slope of Isoquants.

\[
MRTS = -\frac{dK}{dL}
\]

How many units of \( K \) can be reduced to keep \( Q \) constant when we increase \( L \) by one unit. Like \( MRS \), we also have

\[
MRTS = \frac{MP_L}{MP_k}.
\]
2 Long Run Production Function

Figure 3: Long Run Production Function.

Figure 4: $K$ vs $L$, Isoquant Curve.
3 Returns to Scale

Proof. Since $K$ is a function of $L$ on the isoquant curve,

$$q(K(L), L) = 0$$

$$\Rightarrow$$

$$\frac{\partial q}{\partial L} \frac{dK}{dL} + \frac{\partial q}{\partial L} = 0$$

$$\Rightarrow$$

$$\frac{dK}{dL} = \frac{MP_L}{MP_K} .$$

Perfect Substitutes (Inputs). (see Figure 5)

![Isoquant Curve, Perfect Substitutes.](image)

Perfect Complements (Inputs). (see Figure 6)

3 Returns to Scale

Marginal Product of Capital.

$$MP_K = \frac{\partial q(K, L)}{\partial K}$$

Marginal Product of Labor

$$K constant , L \uparrow \rightarrow q?$$
Figure 6: Isoquant Curve, Perfect Complements.

Marginal Product of Capital

$L \text{ constant}$, $K \uparrow \rightarrow q$?

What happens to $q$ when both inputs are increased?

$K \uparrow$, $L \uparrow \rightarrow q$?

Increasing Returns to Scale.

- A production function is said to have increasing returns to scale if
  
  $Q(2K, 2L) > 2Q(K, L),$
  
  or
  
  $Q(aK, aL) = 2Q(K, L), a < 2.$

- One big firm is more efficient than many small firms.
- Isoquants get closer as we move away from the origin (see Figure 7).
Figure 7: Isoquant Curves, Increasing Returns to Scale.