1. Setting up the KT we have:

\[ f(x, \alpha, \beta) + \lambda g(x, \alpha, \beta) \]

Taking the FOC for each \( x_i \) we have:

\[ \frac{\partial f}{\partial x_i} + \lambda \frac{\partial g}{\partial x_i} \equiv 0 \]

Notice that these are identities - i.e., they always hold. Suppose that we are at the optimal \( x^*(a, \beta) \). If we take the derivative of \( M \) with respect to \( \alpha \) we get:

\[ \frac{\partial M}{\partial \alpha} = \frac{\partial f}{\partial \alpha} + \lambda \frac{\partial g}{\partial \alpha} + \sum_{i=1}^{n} \left( \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial \alpha} + \lambda \frac{\partial g}{\partial x_i} \frac{\partial x_i}{\partial \alpha} \right) \]

Using the FOC above and noting that we can factor \( \frac{\partial x_i}{\partial \alpha} \) out of each summation term, all the last terms cancel out and we are left with:

\[ \frac{\partial M}{\partial \alpha} = \frac{\partial f}{\partial \alpha} + \lambda \frac{\partial g}{\partial \alpha} \]

The same thing holds for \( \beta \).

2. For each of the following, derive \( x(p,m), e(p,u), v(p,m), h(p,u) \) using the standard budget constraint \( p_1 x_1 + p_2 x_2 = m \):

(a) The utility function here is strictly concave. Since the budget constraint is linear, we will always end up at a corner. Thus:

\[ x_1(p,m) = \begin{cases} \frac{m}{p_1} & p_1 \leq p_2 \\ 0 & \text{otherwise} \end{cases} \]

\[ x_2(p,m) = \begin{cases} \frac{m}{p_1} & p_1 > p_2 \\ 0 & \text{otherwise} \end{cases} \]

\[ v(p,m) = \begin{cases} m & \min(p_1, p_2) \end{cases} \]

\[ v(p,e(p,u)) = u \rightarrow \frac{e(p,u)}{\min(p_1, p_2)} = u \text{ so:} \]

\[ e(p,m) = u \min(p_1, p_2) \]

\[ h_1(p,u) = \begin{cases} u & p_1 \leq p_2 \\ 0 & \text{otherwise} \end{cases} \]

\[ h_2(p,u) = \begin{cases} u & p_1 > p_2 \\ 0 & \text{otherwise} \end{cases} \]

(b) \( u(x_1, x_2) = \min(x_1, x_2) \)
(c) First argue that the only point that will ever be chosen has \( x_1 = x_2 \):

1. Suppose that \( x_1 > x_2 \), \( \exists \varepsilon \) such that \( u(x_1 - \varepsilon, x_2 + \frac{p_2}{p_1} \varepsilon) > u(x_1, x_2) \). but \( p_1(x_1 - \varepsilon) + p_2(x_2 + \varepsilon) = m \) and is affordable. Thus the agent is not profit maximizing.

2. Now substitute in for \( x_2 \) and solve the simplified problem:

\[
\max x_1 \\
ST : (p_1 + p_2) x_1 = m
\]

3. Solving yields:

\[
x_1(p,m) = x_2(p,m) = \frac{m}{p_1 + p_2} \\
v(p,m) = \frac{m}{p_1 + p_2} \\
e(p,u) = u(p_1 + p_2) \\
h_1(p,u) = h_1(p,u) = u
\]

(d) This problem is done in recitation notes one with minor alterations:

\[
x_1(p,m) = \begin{cases} \frac{m}{p_1} & \frac{p_1}{2} \leq p_2 \\ 0 & \text{otherwise} \end{cases} \\
x_2(p,m) = \begin{cases} \frac{m}{p_1} & \frac{p_1}{2} > p_2 \\ 0 & \text{otherwise} \end{cases} \\
v(p,m) = \frac{m}{\min(\frac{p_1}{2}, p_2)}
\]

\[
e(p,m) = u \min(\frac{p_1}{2}, p_2) \\
h_1(p,u) = \begin{cases} \frac{p_1}{2} & \frac{p_1}{2} \leq p_2 \\ 0 & \text{otherwise} \end{cases} \\
h_2(p,u) = \begin{cases} u & \frac{p_1}{2} > p_2 \\ 0 & \text{otherwise} \end{cases}
\]

Note that this solution is identical to the \( \max(x_1, x_2) \) solution with the slight difference that when the \( \frac{p_1}{2} = p_2 \) any combination of inputs yield the same solution.

(e) Taking the FOC:

\[
\begin{align*}
(1) & : \frac{1}{2} x_1^{-\frac{1}{2}} x_1^{\frac{1}{3}} - \lambda p_1 = 0 \\
(2) & : \frac{1}{3} x_1^{\frac{2}{3}} x_2^{-\frac{2}{3}} - \lambda p_2 = 0 \\
(3) & : p_1 x_1 + p_2 x_2 = 0
\end{align*}
\]

Dividing (1) by (2):

\[
\frac{3 x_2}{2 x_1} = \frac{p_1}{p_2} \rightarrow x_2 = \frac{p_1}{p_2} \frac{2}{3} x_1
\]
Substitution into (3) yields:

\[ p_1 x_1 + p_1 \frac{2}{3} x_1 = m \]

Thus:

\[
\begin{align*}
x_1(p,m) &= \frac{3m}{5p_1} \\
x_2(p,m) &= \frac{2m}{5p_2} \\
v(p,m) &= \left( \frac{3m}{5p_1} \right)^{1/2} \left( \frac{2m}{5p_2} \right)^{1/3} = \left( \frac{m}{5} \right)^{\frac{1}{6}} \left( \frac{3}{p_1} \right)^{\frac{1}{2}} \left( \frac{2}{p_2} \right)^{\frac{1}{3}}
\end{align*}
\]

Inverting to get the expenditure function we have:

\[ u = \left( \frac{e(p,u)}{5} \right)^{\frac{5}{6}} \left( \frac{3}{p_1} \right)^{1/2} \left( \frac{2}{p_1} \right)^{1/3} \rightarrow e(p, u) = 5u^{\frac{6}{5}} \left( \frac{p_1}{3} \right)^{\frac{1}{3}} \left( \frac{p_2}{2} \right)^{\frac{2}{3}} \]

Taking the derivatives wrt \( p_1 \) and \( p_2 \) yields the hicksian demands:

\[
\begin{align*}h_1(p, u) &= 3u^{\frac{6}{5}} \left( \frac{p_1}{3} \right)^{-\frac{2}{5}} \left( \frac{p_2}{2} \right)^{\frac{2}{3}} \\
h_1(p, u) &= 2u^{\frac{6}{5}} \left( \frac{p_1}{3} \right)^{\frac{1}{3}} \left( \frac{p_2}{2} \right)^{-\frac{3}{5}}
\end{align*}
\]

(f) The point of this exercise is to note that we get an identical outcome to problem e. The solution concept is the same.

3. Assuming free disposal we have:

\[ u(l, t, g) = \min(l^2, g^2 + t^2) \]

As we saw in problem 2, a min function requires that the two sides be equal and a linear function requires us to use the input that is cheapest. Thus when \( p_g > p_t \), we have \( l^2 = t^2, g = 0 \). We thus spend \( \frac{1}{2} \) our budget on lime and tonic yielding:

\[
\begin{align*}
l(p, m) &= \left( \frac{m}{p_l + \min(p_g, p_t)} \right)^{\frac{1}{2}} \\
g_1(p, m) &= \left( \frac{m}{p_l + \min(p_g, p_t)} \right)^{\frac{1}{2}} \quad \text{for } p_g \leq p_t \\
g_2(p, m) &= 0 \quad \text{for } p_g > p_t \\
t_2(p, m) &= \left( \frac{m}{p_l + \min(p_g, p_t)} \right)^{\frac{1}{2}} \quad \text{for } p_g > p_t \\
v(p, m) &= \left( \frac{m}{p_l + \min(p_g, p_t)} \right)^{\frac{1}{2}}
\end{align*}
\]
4. Problem 1:

(a) Utility functions are ordinal - this allows us to take monotonic transformations without changing the underlying demand functions.

(b) Transforming the data we have $\tilde{U}(x_1, x_2) = [\ln(U)]^3 = x_1 + \ln(x_2)$

The MRS$_{12} = \frac{\partial U}{\partial x_1} / \frac{\partial U}{\partial x_2} = x_2$. Since for $x_1 = 0$, this is a non infinite amount, we may have the case that $x_1 = 0$.

The MRS$_{21} = \frac{\partial U}{\partial x_2} / \frac{\partial U}{\partial x_1} = \frac{1}{x_2}$. Since for $x_2 = 0$, MRS$_{21} = \infty$, we will never use zero of input 2.

(c) Max$_{x_1, x_2} \tilde{U}(x_1, x_2)$ st $p_1x_1 + p_2x_2 = M$, $x_1 \geq 0$

$L : x_1 + \ln(x_2) + \lambda(M - p_1x_1 - p_2x_2) + \mu x_1$

FOC:

\[
\begin{align*}
1 + \mu & = \lambda p_1 \\
\frac{1}{x_2} & = \lambda p_2 \\
p_1x_1 + p_2x_2 & = M \\
\end{align*}
\]

$x_1 \geq 0, \mu \geq 0, x_1\mu = 0$

Eliminating $\lambda$ we have:

\[
x_2(1 + \mu) = \frac{p_1}{p_2}
\]

so when $x_1 > 0$, $x_2 = \frac{p_1}{p_2}, x_1 = \frac{M}{p_1} - 1$. This will only occur if $M > p_1$

when $\mu > 0$ ($x_1 = 0$), $x_2 = \frac{M}{p_2}$ by the budget constraint.

We thus have:

\[
\begin{align*}
x_1(p_1, p_2, m) & = \frac{m}{p_1} - 1 \quad m < p_1 \quad \text{otherwise} \\
x_2(p_1, p_2, m) & = \frac{m}{p_2} \quad m < p_1 \quad \text{otherwise} \\
V(p_1, p_2, m) & = e^{\left[\ln\left(\frac{m}{p_2}\right)\right]^{\frac{1}{2}}} \quad m < p_1 \quad \text{otherwise}
\end{align*}
\]

(d) The expenditure function $v(p, e(p, u)) = u$. Thus, after some rearranging we have:

\[
e(p, u) = \begin{cases} 
[\ln(u)]^{3}p_2 & m < p_1 \\
\left[\ln(u)]^{3} + 1 - \ln\left(\frac{m}{p_2}\right)\right] & p_1 \quad \text{otherwise}
\end{cases}
\]

5. Consider the indirect utility function given by:

\[
v(p_1, p_2, m) = \frac{m}{p_1 + p_2}
\]
(a) \( x_1(p, m) = \frac{\partial v}{\partial m} = -\frac{m}{(p_1 + p_2)^2} \). Thus:
\[
x_1(p, m) = x_2(p, m) = \frac{m}{p_1 + p_2}
\]
(b) \( v(p_1, p_2, m) = \frac{m}{p_1 + p_2} \rightarrow u = v(p_1, p_2, e(p, u)) = \frac{e(p, u)}{p_1 + p_2} \). Thus:
\[
e(p, u) = u(p_1 + p_2)
\]
(c) To find a representation of the utility function we solve:
\[
\min_{x_1, x_2} \frac{m}{p_1 + p_2}
\]
ST : \( x_1p_1 + x_2p_2 = m \)

The FOC are:
\[
(1) : -\frac{m}{(p_1 + p_2)^2} + \lambda x_1 = 0
\]
\[
(2) : -\frac{m}{(p_1 + p_2)^2} + \lambda x_2 = 0
\]
\[
(3) : x_1p_1 + x_2p_2 = m
\]

From (1) and (2), \( x_1 = x_2 \). Thus one utility function that can satisfy this is \( u(x_1, x_2) = \min(x_1, x_2) \)

6. *Consider the utility function:
\[
u(x_1, x_2) = \min(2x_1 + x_2, x_1 + 2x_2)
\]

(a) The indifference curve will be the NE boundary of the two lines.
(b) The slope of a budget line is \( -\frac{p_1}{p_2} \). If the budget line is steeper than 2, \( x_1 = 0 \). Thus \( x_1 = 0 \) if \( \frac{p_1}{p_2} > 2 \).
(c) This is identical to b: if \( \frac{p_1}{p_2} < \frac{1}{2} \), \( x_2 = 0 \)
(d) If the optimum is unique and on the interior, it must be that \( x_1 + 2x_2 = 2x_1 + x_2 \rightarrow x_1 = x_2 \rightarrow \frac{x_1}{x_2} = 1 \).

(a) Suppose you have no data:
1. incomparable
2. Bundle 1 \( \gtrsim \) Bundle 2
(b) Suppose that you observe that when \( p_1 = 1, p_2 = 1, m = 10 \) the consumer chooses \( x_1 = 2, x_2 = 8 \)
1. Bundle 1 \( \lesssim \) Bundle 2
2. Bundle 1 \( \not\lesssim \) Bundle 2 (note that this one isn’t strict)
(c) Suppose that we have two observations. When \( p_1 = 1, p_2 = 1, m = 10 \) the consumer chooses \( x_1 = 2, x_2 = 8 \). When \( p_1 = 1, p_2 = 3, m = 15 \) the consumer chooses \( x_1 = 15, x_2 = 0 \)
1. Bundle 1 \( \gtrsim \) Bundle 2
2. incomparable