14.05 Intermediate Macro
Pset 4

Due April 19th

Problem 1: Learning by Doing with Spillovers

Consider the model of learning by doing with spillovers (Arrow & Romer) presented in class and assume that the production function is Cobb-Douglas, that is,

\[ Y_t^m = (K_t^m)\alpha (h_t L_t^m)^{1-\alpha} \]

However, assume there are diminishing returns to technological progress, \( h_t = \eta k_t^{\gamma} \), for some constants \( \eta > 0, \ 0 < \gamma < 1 \), where \( k_t = \frac{K_t^m}{L_t^m} \).

i. We want to write the equilibrium dynamics are functions of \( c \) and \( k \) alone:

(a) Express the return \( R \) that firms are willing to pay in equilibrium as a function of \( k_t \) alone.

(b) Express the resource constraint in terms of \( c \) and \( k \).

ii. Imagine the continuous time version of the dynamics in part (a) and draw the phase diagram.

iii. Repeat parts (a) and (b) for the social planner’s problem (Hint: this is similar to the Ramsey model).

iv. How does the phase diagram of part (c) compare to that of part (b)? which line changes, the \( \dot{c} = 0 \) locus or the \( \dot{k} = 0 \) locus? What happens to the steady state levels of \( c \) and \( k \)?

v. If the equilibrium allocations differ from the planner’s allocations, describe a policy that would restore efficiency.
Problem 2: Tax smoothing

Consider a two-period economy. Households preferences are given by
\[ U = u(c_1, c_2, n_1, n_2) = c_1 - n_1^2 + \beta (c_2 - n_2^2), \]
where \( c_t \geq 0 \) is consumption in period \( t \in \{1, 2\} \) and \( n_t \geq 0 \) is labor supply. Labor is used to produce output with the technology \( y_t = An_t \) (there is no capital). The wage is thus given by \( w_t = A \), for \( t \in \{1, 2\} \). The government taxes labor income at rates \( \tau_t \) in period \( t \), so households' intertemporal budget constraint is given by
\[ c_1 + \frac{1}{1+r} c_2 = (1 - \tau_1) An_1 + \frac{1}{1+r} (1 - \tau_2) An_2 \]
The government has constant expenditures, \( g_t = g \) for \( t \in \{1, 2\} \). Its intertemporal budget constraint is thus given by
\[ IBC \equiv (\tau_1 An_1 - g_1) + \frac{1}{1+r} (\tau_2 An_2 - g_2) = 0 \]
Finally, the resource constraints in the economy are \( y_1 = An_1 = c_1 + g \) and \( y_2 = An_2 = c_2 + g \).

1) Consider the household's optimal consumption and labor-supply problem. Argue that the solution is interior only if the interest rate \( r \) is such that \( \frac{1}{1+r} = \beta \). Assume that this is the case for the rest of the exercise.

2) Solve for the household's optimal \( n_1 \) and \( n_2 \) as functions of \( \tau_1 \) and \( \tau_2 \).

3) Use the two resource constraints to replace \( c_t = An_t - g \) into \( U \). Next, use the previous result to replace \( n_t \) with a function of \( \tau_t \). You should now have expressed the household's utility \( U \) as a function of the two tax rates:
\[ U = U(\tau_1, \tau_2) \]

4) Do the same for the government's intertemporal budget: replace \( n_t \) with the function of \( \tau_t \) that you found in part 2 so as to express \( IBC \) in terms of \( \tau_1 \) and \( \tau_2 \):
\[ IBC = IBC(\tau_1, \tau_2) \]

5) It follows that the optimal policy is given by the combination of \( \tau_1 \) and \( \tau_2 \) that solves the following problem:
\[ \max U(\tau_1, \tau_2) \]
\[ s.t. \quad IBC(\tau_1, \tau_2) = 0 \]
Prove that the optimal policy satisfies \( \tau_1 = \tau_2 \) (tax smoothing).

6) Suppose that we increase \( g_1 \) but reduce \( g_2 \) so that \( g_1 + \beta g_2 \) stays constant. What happens to the optimal taxes? Explain.