Lecture 14
Infinitely Repeated Games II

14.12 Game Theory
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Road Map

1. Folk Theorem
2. Applications (Problems)
Folk Theorem

**Definition:** $v = (v_1, v_2, ..., v_n)$ is feasible iff $v$ is a convex combination of pure-strategy payoff-vectors:

$$v = p_1 u(a_1) + p_2 u(a_2) + \ldots + p_m u(a_m),$$

where $p_1 + p_2 + \ldots + p_m = 1$, and $u(a_i)$ is the payoff vector at strategy profile $a_i$ of the stage game.

**Theorem:** Let $x = (x_1, x_2, ..., x_n)$ be a feasible payoff vector, and $e = (e_1, e_2, ..., e_n)$ be a payoff vector at some equilibrium of the stage game such that $x_i > e_i$ for each $i$. Then, there exist $\delta < 1$ and a strategy profile $s$ such that $s$ yields $x$ as the expected average-payoff vector and is a SPE whenever $\delta > \delta$. 
Folk Theorem in PD

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<tr>
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<th>C</th>
<th>D</th>
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<tbody>
<tr>
<td>C</td>
<td>5.5</td>
<td>0.6</td>
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<tr>
<td>D</td>
<td>6.0</td>
<td>1.1</td>
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- A SPE with PV $(1.1, 1.1)$?
  - With PV $(1.1, 5)$?
  - With PV $(6, 0)$?
  - With PV $(5.9, 0.1)$?
Proof for a special case

• Assume $x = u(a^*) = (u_1(a^*), \ldots, u_n(a^*))$ for some $a^*$.
• $s^*$: Every player $i$ plays $a_i^*$ until somebody deviates and plays $e_i$ thereafter.
• Average value of $i$ from $s^*$ is $x_i = u_i(a^*)$.
• $s^*$ is a SPE $\iff \delta \geq \bar{\delta}$ where

$$\bar{\delta} = \max_{i, a_i} \frac{u_i(a_i, a_{-i}^*) - x_i}{u_i(a_i, a_{-i}^*) - e_i} < 1$$
Applications/Problems
2010 Midterm 2, P2

Diagram showing decision paths and payoffs for Aliza and Colin.
Range of $\delta$ for SPE

- Alice Hires and Bob and Colin both Work until any of the workers Shirk; Alice Hires and Bob and Colin both Shirk thereafter.
- Alice Always Hires. Both workers Work at $t = 0$. At any $t > 0$, each worker Works if the previous play is (Hire, Work, Work) or (Hire, Shirk, Shirk); each worker Shirks otherwise.
2007 Midterm 2, P3

• Stage Game: Linear Bertrand Duopoly (c=0; Q=1-p)
• s*: They both charge 1/2 until somebody deviates; they both charge 0 thereafter.
• s**: n + 1 modes: Collusion, W1, W2, ..., Wn. Game starts at Collusion. Both charge 1/2 in the Collusion mode and p*<1/2 in W1,. . . , Wn. Without deviation, Collusion leads to Collusion, W1 leads to W2,. . . , Wn-1 leads to Wn, and Wn leads to Collusion. Any deviation leads to W1.