Handout 2 on inefficiency with incomplete markets

I. Change in production.
Trading in each state of nature - no trade across states - production decision before state is known

Consumer choice for type A

\[
\begin{align*}
\max & \sum_s \pi_s u^A(s, x_{1s}) \\
\text{s.t.} & \quad x_{0s} + p_s x_{1s} = e_0, \quad s = 1, 2
\end{align*}
\]

\[\pi_s u^A_0(s, x_{1s}) = \lambda^A_s; \quad \pi_s u^A_1(s, x_{1s}) = \lambda^A_s p_s\]  

By Roy's identity, we have

\[\frac{d \sum_s \pi_s u^A(s, x_{1s})}{dp_s} = -\pi_s u^A_0(s, x_{1s}) x_{1s}\]

Consumer/producer choice for type B

\[
\begin{align*}
\max & \sum_s \pi_s u^B(s, x_{1s}) \\
\text{s.t.} & \quad x_{0s} + p_s x_{1s} = e_0 + p_s e_{1s}, \quad s = 1, 2 \\
& \quad F(e_{11}, e_{12}) = 0
\end{align*}
\]

\[\frac{F_1}{F_2} = \frac{\lambda^B_1 p_1}{\lambda^B_2 p_2} = \frac{\pi_1 u_0^B(s, x_{11}) p_1}{\pi_2 u_0^B(s, x_{12}) p_2} = \frac{\pi_1 u_1^B(s, x_{11})}{\pi_2 u_1^B(s, x_{12})} = \frac{\pi_1 u_0^B}{\pi_2 u_1^B}\]
Market clearance

\[ x_1^A(p_s, e_0^A) + x_1^B(p_s, e_0^B + p_s e_{1s}^B) = e_{1s}^B \]  \hspace{1cm} (6)

implying:

\[ p_s = p(e_{1s}^B) \]  \hspace{1cm} (7)

Impact of deviation from production decision

\[ \frac{de_{12}}{de_{11}} = -\frac{F_1}{F_2} \]  \hspace{1cm} (8)

\[ x_{1s}^A = -(x_{1s}^B - e_{1s}^B) \]  \hspace{1cm} (9)

\[
\frac{d}{de_{11}^B} \sum_s \pi_s u^A(s) = -\pi_1 u_0^B (1) x_{11}^A p'(e_{11}^B) - \pi_2 u_0^A (2) x_{12}^A p'(e_{12}^B) \frac{de_{12}}{de_{11}} \\
= -\pi_1 u_0^B (1) \left[ x_{11}^A p'(e_{11}^B) + \frac{\pi_2 u_0^B (2)}{\pi_1 u_0^A (1)} x_{12}^A p'(e_{12}^B) \frac{de_{12}}{de_{11}} \right] \]  \hspace{1cm} (10)

\[
\frac{d}{de_{11}^B} \sum_s \pi_s u^B(s) = -\pi_1 u_0^B (1) \left[ (x_{11}^B - e_{11}^B) p'(e_{11}^B) + \frac{\pi_2 u_0^B (2)}{\pi_1 u_0^B (1)} (x_{12}^B - e_{12}^B) p'(e_{12}^B) \frac{de_{12}}{de_{11}} \right] \\
= \pi_1 u_0^B (1) \left[ x_{11}^A p'(e_{11}^B) + \frac{\pi_2 u_0^B (2)}{\pi_1 u_0^B (1)} x_{12}^A p'(e_{12}^B) \frac{de_{12}}{de_{11}} \right] \]  \hspace{1cm} (11)
II. Change in production with redistribution
We now add redistribution in numeraire good, at the same level in both states of nature.

This changes market clearance to:

\[ x_1^A (p_s^A, e_0^A - T) + x_1^B (p_s^B, e_0^B + T) = e_{1s}^B \] \hspace{1cm} (12)

implying:

\[ p_s = p(e_{1s}^B, T) \] \hspace{1cm} (13)

Note that

\[
\frac{\partial p_s}{\partial T} = \frac{\partial x_1^A}{\partial I} - \frac{\partial x_1^B}{\partial I} - \frac{\partial x_1^A}{\partial p} \frac{\partial I}{\partial I} + \frac{\partial x_1^B}{\partial p} \frac{\partial I}{\partial I} + \left( e_{1s}^B + T_1 \right) \frac{\partial x_1^B}{\partial I} \] \hspace{1cm} (14)

\[
\frac{\partial p_s}{\partial T_1} = p_s \left( \frac{\partial x_1^A}{\partial I} - \frac{\partial x_1^B}{\partial I} \right) \] \hspace{1cm} (15)

As long as the income derivatives of A and B are different, these are nonzero. Also the demand derivatives are evaluated at different prices and incomes in the different states.
Starting with zero transfers, consider a derivative change in the two transfers, satisfying (for some constant $k$).

\[ dT_1 = k dT_0 \]  \hspace{1cm} (16)

This implies that

\[ \frac{dp_s}{dT_0} \equiv \frac{\partial p_s}{\partial T_0} + k \frac{\partial p_s}{\partial T_1} = \frac{(1 + kp_s)}{\partial I} \left[ \frac{\partial x_1^A}{\partial I} - \frac{\partial x_1^B}{\partial I} \right] \]

\[ = (1 + kp_s) \alpha_s \]  \hspace{1cm} (17)

We want to evaluate the impact of a redistribution on expected utilities in equilibrium.

\[ \frac{d}{dT_0} \sum_s \pi_s u^A (s) = - \sum_s \pi_s \left( u_0^A (s) + ku_i^A (s) \right) \]

\[ = -\pi_i u_0^A (1) x_{11}^A \frac{dp_1}{dT_0} - \pi_2 u_0^A (2) x_{12}^A \frac{dp_2}{dT_0} \]

\[ = -\pi_i u_0^A (1) \left[ (1 + kp_1) (1 + x_{11}^A) + \frac{\pi_2 u_0^A (2)}{\pi_1 u_0^A (1)} (1 + kp_2) (1 + x_{12}^A) \right] \]  \hspace{1cm} (18)

Similarly, using the same substitutions as in (11),

\[ \frac{d}{dT_0} \sum_s \pi_s u^B (s) = \pi_i u_0^B (1) \left[ (1 + kp_1) (1 + x_{11}^A) + \frac{\pi_2 u_0^B (2)}{\pi_1 u_0^B (1)} (1 + kp_2) (1 + x_{12}^A) \right] \]  \hspace{1cm} (19)
Generically we have different prices and demands in the two states and different marginal rates of substitution for the two agents. The aim is to find a constant, \( k \), so that the changes in transfers leave both of them better off or both worse off (in which case we reverse the direction of transfers). This may be possible – this model does not fit the Inefficiency Theorem. Contrasting (18) and (19) to (10) and (11), we have an extra degree of freedom in seeking a Pareto gain.

For a Pareto gain, we need to find a value of \( k \) such that (18) and (19) are both positive or both negative (calling for a reversal of the direction of redistribution). This requires

\[
\frac{\pi_2 u_0^a (2)}{\pi_1 u_0^a (1)} < \frac{1 + k p_2}{1 + k p_1} \left( \frac{1 + x_{i2}^d}{1 + x_{i1}^d} \right) < \frac{\pi_2 u_0^b (2)}{\pi_1 u_0^b (1)} \tag{20}
\]

or

\[
\frac{\pi_2 u_0^a (2)}{\pi_1 u_0^a (1)} > \frac{1 + k p_2}{1 + k p_1} \left( \frac{1 + x_{i2}^d}{1 + x_{i1}^d} \right) > \frac{\pi_2 u_0^b (2)}{\pi_1 u_0^b (1)} \tag{21}
\]