14.123 Problem Set 1 Solution
Suehyun Kwon

Q1. Let \( P \) be the set of all lotteries \( p = (p_x, p_y, p_z) \) on a set \( C = \{x, y, z\} \) of consequences. Below, you are given pairs of indifference sets on \( P \). For each pair, check whether the indifference sets belong to a preference relation that has a Von-Neumann and Morgenstern representation (i.e. expected utility representation). If the answer is Yes, provide a Von-Neumann and Morgenstern utility function; otherwise show which Von-Neumann and Morgenstern axiom is violated. (In the figures below, setting \( p_z = 1 - p_x - p_y \), we describe \( P \) as a subset of \( \mathbb{R}^2 \)).

(a) \( I_1 = \{p|1/2 \leq p_y \leq 3/4\} \) and \( I_2 = \{p|p_y = 1/4\} \):

No, the Independence Axiom is violated. I’ll use (2.2) from Question 2. Take \((1/4, 3/4), (1/2, 1/2) \in I_1\) and \(a = 2\). From \((1/4, 3/4) \sim (1/2, 1/2)\), we have

\[
(1/4, 3/4) = 2(1/4, 3/4) + (-1)(1/4, 3/4)
\]

\[
\sim 2(1/2, 1/2) + (-1)(1/4, 3/4) = (3/4, 1/4),
\]

which is a contradiction to \((3/4, 1/4) \in I_2\).

(b) \( I_1 = \{p|p_y = p_x\} \) and \( I_2 = \{p|p_y = p_x + 1/2\} \):

Yes, an example is \( U(p) = p_x - p_y \).

Q2. For any preference relation that satisfies the Independence Axiom, show that the following are true.

(a) For any \( p, q, r, r' \in P \) with \( r \sim r' \) and any \( a \in (0, 1] \),

\[
ap + (1 - a)r \succeq aq + (1 - a)r' \iff p \succeq q.
\]

\( r \sim r' \) implies that \( r \succeq r' \) and \( r' \succeq r \). From the Independence Axiom, for any \( a \in (0, 1] \),

\[
p \succeq q \iff ap + (1 - a)r \succeq aq + (1 - a)r.
\]

The Independence Axiom also implies that

\[
aq + (1 - a)r \succeq aq + (1 - a)r',
\]

\[
aq + (1 - a)r' \succeq aq + (1 - a)r,
\]
and we have

\[ \begin{align*}
ap + (1 - a)r & \geq aq + (1 - a)r \implies ap + (1 - a)r \geq aq + (1 - a)r', \\
ap + (1 - a)r & \geq aq + (1 - a)r' \implies ap + (1 - a)r \geq aq + (1 - a)r
\end{align*} \]

by transitivity.

\[ p \geq q \iff ap + (1 - a)r \geq aq + (1 - a)r'. \]

(b) For any \( p, q, r \in P \) and any real number \( a \) such that \( ap + (1 - a)r, aq + (1 - a)r \in P \),

if \( p \sim q \), then \( ap + (1 - a)r \sim aq + (1 - a)r \).

The case \( a \in (0,1] \) is given by the Independence Axiom, and the case \( a = 0 \) always holds from \( r \sim r \).

For \( a > 1, 1/a \in [0,1] \), and the Independence Axiom gives that

\[ \begin{align*}
ap + (1 - a)r & \sim aq + (1 - a)r \\
\iff & \frac{1}{a}(ap + (1 - a)r) + \frac{a - 1}{a} - r \sim \frac{1}{a}(aq + (1 - a)r) + \frac{a - 1}{a} - r \\
\iff & p \sim q.
\end{align*} \]

For \( a < 0, 1/(1 - a) \in \mathbb{R} \) such that \( ap + (1 - a)r, aq + (1 - a)r \in P \),

if \( p \sim q \), then \( ap + (1 - a)r \sim aq + (1 - a)r \).

The case \( a \in (0,1] \) is given by the Independence Axiom, and the case \( a = 0 \) always holds from \( r \sim r \).

(c) For any \( p, q \in P \) with \( p > q \) and any \( a, b \in [0,1] \) with \( a > b \),

\[ ap + (1 - a)q > bp + (1 - b)q. \]
If $b = 0$, the Independence Axiom gives that
\[ ap + (1 - a)q \succ aq + (1 - a)q \sim q. \]

For $b > 0$, we have $b/a \in (0, 1)$, and
\[
\begin{align*}
ap + (1 - a)q & \succ q \\
\implies ap + (1 - a)q & \sim \frac{b}{a}(ap + (1 - a)q) + \frac{a - b}{a}(ap + (1 - a)q) \\
& \succ \frac{b}{a}(ap + (1 - a)q) + \frac{a - b}{a}q \sim bp + (1 - b)q.
\end{align*}
\]

(d) There exist $c^B, c^W \in C$ such that for any $p \in P$,
\[ c^B \succeq p \succeq c^W. \tag{4} \]

[Hint: use completeness and transitivity to find $c^B, c^W \in C$ with $c^B \succeq c \succeq c^W$ for all $c \in C$; then use induction on the number of consequences and the Independence Axiom.]

The set of consequences $C$ is finite. Let $n$ be the number of consequences. When $n = 1$, $c^B \sim p \sim c^W$ for all $p \in P$.

Suppose that for $n = k$, there exist $c^B, c^W \in C$ such that for any $p \in P$,
\[ c^B \succeq p \succeq c^W. \tag{*} \]

Consider $n = k + 1$. Let $C = \{c_1, \ldots, c_{k+1}\}$ and $C' = \{c_1, \ldots, c_k\}$. From (*), there exist $c^{B'}, c^{W'} \in C'$ such that $c^{B'} \succeq p' \succeq c^{W'}$. If $c_{k+1} \succ c^{B'}$, let $c^B = c_{k+1}, c^W = c^{W'}$. If $c^{W'} \succ c_{k+1}$, let $c^B = c^{B'}, c^W = c_{k+1}$. Otherwise, $c^B = c^{B'}, c^W = c^{W'}$. Any $p \in P$ can be written as $p = ap' + (1 - a)c_{k+1}$ for some $a \in [0, 1]$ and a lottery $p'$ over $C' = \{c_1, \ldots, c_k\}$.

We have $c^B \succeq p', c_{k+1} \succeq c^W$, and by the Independence Axiom,
\[
\begin{align*}
c^B & \succeq ap' + (1 - a)c^B \\
& \succeq ap' + (1 - a)c_{k+1} = p \\
& \succeq ap' + (1 - a)c^W \\
& \succeq c^W.
\end{align*}
\]

Q3. Let $P$ be the set of probability distribution on $C = \{x, y, z\}$. Find a continuous preference relation $\succeq$ on $P$, such that the indifference sets are all straight lines, but $\succeq$ does not have a von Neumann-Morgenstern utility representation.
Consider a preference relation represented by the following utility function

\[ U(p_x, p_y, p_z) = \frac{p_y}{2 - p_x}. \]

\( \succeq \) is complete, transitive and continuous, and the indifference set are straight lines, but the Independence Axiom is not satisfied.

Q4. Let \( \succeq \) be the "at least as likely as" relation defined between events in Lecture 3. Show that \( \succeq \) is a qualitative probability.

From \( P1 \), \( \succeq \) is a preference relation, which implies that it’s complete and transitive.

The second part follows from

\[
\begin{align*}
B \succeq C & \iff f_B^{x,x'} \succeq f_C^{x,x'} \text{ for some } x, x' \in C, x \succ x' \\
& \iff f_B^{x,x'} \succeq f_{C \cup D}^{x,x'} \\
& \iff B \cup D \succeq C \cup D. \tag{\cdot: P2}
\end{align*}
\]

Lastly, from \( P4 \), there exists \( x, x' \in C \) with \( x \succ x' \). For any event \( B \),

\[
\begin{align*}
x \succ x' & \implies f_B^{x,x'} \succeq f_{\emptyset}^{x,x'} \\
& \implies B \succeq \emptyset. \tag{\cdot: P2}
\end{align*}
\]

Given any \( x, x' \in C \) with \( x \succ x' \), we have \( f_S^{x,x'} \succeq f_{\emptyset}^{x,x'} \) from \( P2 \). There exist no \( x, x' \in C \) with \( x \succ x' \) such that \( f_{\emptyset}^{x,x'} \succeq f_S^{x,x'} \).

\[
S \succeq \emptyset, \emptyset \notin S \implies S \succ \emptyset.
\]
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