14.123 Microeconomic Theory III. 2014

Problem Set 1. Solution.

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1. See solution by Suehyun Kwon of Problem Set 1, 2010, Question 2.

2.

2.1 i. Yes. For example, \( u(x) = 3, u(y) = 2, u(z) = 0 \).

ii. No. Suppose that such expected utility representation exists. Without loss of generality, normalize \( u(z) = 0 \) and from the ordering of the lotteries it follows

\[
\frac{u(x)}{4} + \frac{u(y)}{4} > \frac{3u(x)}{4} > \frac{5u(x)}{6} + \frac{u(y)}{6} > \frac{u(x)}{2} + \frac{u(y)}{3}.
\]

From the first inequality, \(-2u(x) + u(y) > 0\), from the third inequality \(2u(x) - u(y) > 0\) which contradicts the first inequality.

2.2 i. Yes, take \( u(x) = 2, u(y) = 1, u(z) = 0 \).

ii. No. Consider \( p = (1, 0, 0), q = (0, 0, 1) \) which are equivalent and consider a half-half mixture of them with \( r = (0, 1, 0) \). Then \( p' = (1/2, 1/2, 0) \) and \( q' = (0, 1/2, 1/2) \) lie on the different indifference sets, which contradicts IA.

iii. No, as the indifference sets are not straight lines which contradicts IA.

2.3 Consider lexicographic preferences: \( p \succ q \) if and only if \( p(x) > q(x) \) or \( p(x) = q(x) \) and \( p(y) > q(y) \). Since this preference is discontinuous, there is no representation, let alone expected utility representation.

3. In this question I refer to the second condition in Definition 3.2 in Lecture notes as (\*). I will also use the following consequence of (\*).

**Claim.** Consider \( B, C, D \in A \) such that \( D \subseteq B \cap C, D \subseteq B \cap C \). Then \( B \preceq C \iff B \setminus D \succeq C \setminus D \).

I refer to this claim by (**). To see that it is true, observe that \( B = (B \setminus D) \cup D \) and \( C = (C \setminus D) \cup D \). Since \((B \setminus D) \cap D = \emptyset\) and \((C \setminus D) \cap D = \emptyset\), it follows by (\*) that \( B \preceq C \iff B \setminus D \succeq C \setminus D \).
3.1 Let $X = A_1 \cap B_2$, $A'_1 = A_1 \setminus X$, $B'_2 = B_2 \setminus X$. By (**), $A_1 \cup A_2 \succeq B_1 \cup B_2 \iff A'_1 \cup A_2 \succeq B_1 \cup B'_2$. Then

$$A'_1 \cup A_2 \succeq A'_1 \cup B_2 = A_1 \cup B'_2 \succeq (B_1 \setminus B'_2) \cup B'_2 = B_1 \cup B'_2$$

where I used twice (*) to get inequalities and equalities are simple set manipulations.

3.2 Denote by $\succeq$ preference relation “$\succeq$ given $D$”. Completeness and transitivity of $\succeq$ follow from completeness and transitivity of $\succ$. Consider $B, C, E \in \mathcal{A}$ such that $B \cap E = C \cap E = \emptyset$. Then condition 2 in the definition of the qualitative probability is obtained by the following line of inequalities.

$$B \succeq C \iff B \cap D \succeq C \cap D \iff (B \cap D) \cup E \succeq (C \cap D) \cup E \iff$$

$$\quad (B \cap D) \cup (E \cap D) \cup (E \setminus D) \succeq (C \cap D) \cup (E \cap D) \cup (E \setminus D) \iff$$

$$\quad (B \cap D) \cup (E \cap D) \succeq (C \cap D) \cup (E \cap D) \iff B \cup E \succeq C \cup E,$$

where I use (*) and set manipulations to obtain the equivalence relation. $B \succeq \emptyset$ follows from the corresponding property of $\succeq$, and $S \succ \emptyset$ follows from $D$ being non-null.

3.3 Suppose to contradiction that $A_1 \succ B_1$. By transitivity of $\sim$, for all $1 \leq i \leq n$, $A_i \succ B_i$. By the argument as in part 3.1 of this question, it is possible to show that $A_1 \cup A_2 \succeq B_1 \cup B_2$ and iteratively applying this inequality I get that $S = \bigcup_{i=1}^{n} A_i \succeq \bigcup_{i=1}^{n} B_i = S$, contradiction.