Problem Set 1 - Solutions

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Question (i)

Part (a)
Set $\bar{X} = \sum_{i=1}^{n} X_i$, aggregate endowment. From the lecture notes, an optimal risk sharing contract $(Y_1, \ldots, Y_n)$ is given by

$$Y_i = \frac{1/\alpha_i}{1/\alpha_1 + \ldots + 1/\alpha_n} \bar{X} + \tau_i \quad \forall i = 1, \ldots, n,$$

where $\tau_1, \ldots, \tau_n \in \mathbb{R}$ are state-independent transfers such that $\sum_{i=1}^{n} \tau_i = 0$. Agent $i$ gets

$$E[u_i(Y_i)] = -E \left[ e^{-\frac{1}{\alpha_i + \ldots + 1/\alpha_n} \bar{X}} \right] e^{-\alpha_i \tau_i} = -e^{-\alpha_i \left( \frac{1/\alpha_i}{1/\alpha_1 + \ldots + 1/\alpha_n} E[\bar{X}] - \frac{1/\alpha_i}{2(1/\alpha_1 + \ldots + 1/\alpha_n)^2} Var(\bar{X}) + \tau_i \right)}.$$

Hence

$$Y_i \sim \frac{1/\alpha_i}{1/\alpha_1 + \ldots + 1/\alpha_n} E[\bar{X}] - \frac{1/\alpha_i}{2(1/\alpha_1 + \ldots + 1/\alpha_n)^2} Var(\bar{X}) + \tau_i,$$

where the right-hand side is the certainty equivalent of $Y_i$ for agent $i$.

Part (b)
Notice that $E[\bar{X}] = n \mu$ and

$$Var(\bar{X}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Cov}(X_i, X_j) = \sum_{i=1}^{n} Var(X_i) + \sum_{i=1}^{n} \sum_{j \neq i} \text{Cov}(X_i, X_j) = n \sigma^2 (1 + (n-1)\rho).$$

The results in Part (a) become

$$Y_i = \frac{1}{n} \bar{X} + \tau_i,$$

$$E[u_i(Y_i)] = -e^{-\alpha \left( \frac{1}{n} E[\bar{X}] - \frac{\sigma^2}{2n} Var(\bar{X}) + \tau_i \right)} = -e^{-\alpha (\mu - \frac{\sigma^2}{2n} (1 + (n-1)\rho) + \tau_i)},$$

$$Y_i \sim \mu - \frac{\alpha \sigma^2}{2n} (1 + (n-1)\rho) + \tau_i.$$
Summing up the agents’ certainty equivalent, the society as a whole is willing to pay
\[ \eta \mu - \frac{\alpha}{2} \sigma^2 (1 + (n - 1)\rho) \]
for the assets. If we focus on symmetric contracts, \( \tau_1 = \ldots = \tau_n = 0 \). Therefore
\[ (\sigma^2, \rho) \preceq (\tilde{\sigma}^2, \tilde{\rho}) \iff \sigma^2 (1 + (n - 1)\rho) \leq \tilde{\sigma}^2 (1 + (n - 1)\tilde{\rho}), \]
which is the agent’s preference over the assets. Comments: (i) fixing \( \rho \), the agent prefers lower \( \sigma^2 \), and (ii) fixing \( \sigma^2 \), the agents prefers lower \( \rho \).^1 Intuition: \( Var(X) \) (and therefore \( Var(Y_i) \)) is decreasing in \( \sigma^2 \) and \( \rho \), and the agent is risk averse.

**Question (ii)**

See solution to question 2, pset 1 2010.

**Question (iii)**

See solution to question 2, pset 1 2014.

**Question (iv)**

See solution to question 2, final 2011.

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^1 Notice that \( 1 + (n - 1)\rho \geq 0 \) by assumption, otherwise the assets are not jointly normal, that is, \( \Sigma \) is not positive semi-definite. In fact, \( 1 + (n - 1)\rho \) is an eigenvalue of \( \Sigma \).