Q1. There are two urns, A and B, each consisting of 100 balls, some are black and some are red. In urn A there are 30 red balls, but the number of red balls in urn B is not known. We draw a ball from urn A with color \( \alpha \) and a ball from urn B with color \( \beta \). Consider the following acts:

\[
\begin{align*}
  f_{A,r} &= \begin{cases} 
    100 & \text{if } \alpha = \text{red} \\
    0 & \text{if } \alpha = \text{black}
  \end{cases} \\
  f_{A,b} &= \begin{cases} 
    0 & \text{if } \alpha = \text{red} \\
    100 & \text{if } \alpha = \text{black}
  \end{cases} \\
  f_{B,r} &= \begin{cases} 
    110 & \text{if } \beta = \text{red} \\
    0 & \text{if } \beta = \text{black}
  \end{cases} \\
  f_{B,b} &= \begin{cases} 
    0 & \text{if } \beta = \text{red} \\
    110 & \text{if } \beta = \text{black}
  \end{cases}
\end{align*}
\]

Let \( c \) be the choice function induced by \( \succeq \). Find the sets \( c(\{ f_{A,r}, f_{A,b}, f_{B,r}, f_{B,b} \}) \) that are consistent with \( 110 > 100 > 0 \) and Savages postulates.

In the terminology of Lecture 3, the set of states is

\[
\{(\alpha, \beta)\} = \{(r, r), (r, b), (b, r), (b, b)\},
\]

and the set of consequences is \( C = \{0, 100, 110\} \). Under Savage’s postulates, there exists a utility function \( u : C \to \mathbb{R} \) and a probability measure \( p : 2^S \to [0, 1] \) such that

\[
f \succeq g \iff \sum_{c \in C} p(\{s | f(s) = c\})u(c) \geq \sum_{c \in C} p(\{s | g(s) = c\})u(c).
\]

From \( 110 > 100 > 0 \), we have \( u(110) > u(100) > u(0) \). For any probability measure \( p \), we have

\[
0.7 \ast u(100) + 0.3 \ast u(0) = \sum_{c \in C} p(\{s | f_{A,b}(s) = c\})u(c) \\
> \sum_{c \in C} p(\{s | f_{A,r}(s) = c\})u(c) = 0.3 \ast u(100) + 0.7 \ast u(0),
\]

and \( f_{A,b} \succ f_{A,r} \).

On the other hand, the expected utility from \( f_{B,r} \) and \( f_{B,b} \) are

\[
\sum_{c \in C} p(\{s | f_{B,r}(s) = c\})u(c) = p(\beta = \text{red})u(110) + p(\beta = \text{black})u(0),
\]

\[
\sum_{c \in C} p(\{s | f_{B,b}(s) = c\})u(c) = p(\beta = \text{red})u(0) + p(\beta = \text{black})u(110),
\]

and \( f_{A,b} \succ f_{A,r} \).

On the other hand, the expected utility from \( f_{B,r} \) and \( f_{B,b} \) are

\[
\sum_{c \in C} p(\{s | f_{B,r}(s) = c\})u(c) = p(\beta = \text{red})u(110) + p(\beta = \text{black})u(0),
\]

\[
\sum_{c \in C} p(\{s | f_{B,b}(s) = c\})u(c) = p(\beta = \text{red})u(0) + p(\beta = \text{black})u(110),
\]
respectively.

The preference among \( f_{A,b}, f_{B,r}, f_{B,b} \) depends on the probability measure and the utility function, and the possible choice sets \( c\{f_{A,r}, f_{A,b}, f_{B,r}, f_{B,b}\} \) are

\[
\{f_{A,b}\}, \{f_{B,r}\}, \{f_{B,b}\}, \{f_{A,b}, f_{B,r}\}, \{f_{B,r}, f_{B,b}\}, \{f_{A,b}, f_{B,r}, f_{B,b}\}.
\]

The following is the example of \( p \) and \( u \) for each choice set:

\[
\begin{align*}
\{f_{A,b}\} & : u(0) = 0, u(100) = 0.9, u(110) = 1, p = 0.5 \\
\{f_{B,r}\} & : u(0) = 0, u(100) = 0.9, u(110) = 1, p = 0.7 \\
\{f_{B,b}\} & : u(0) = 0, u(100) = 0.9, u(110) = 1, p = 0.3 \\
\{f_{A,b}, f_{B,r}\} & : u(0) = 0, u(100) = 0.9, u(110) = 1, p = 0.63 \\
\{f_{A,b}, f_{B,b}\} & : u(0) = 0, u(100) = 0.9, u(110) = 1, p = 0.37 \\
\{f_{B,r}, f_{B,b}\} & : u(0) = 0, u(100) = 0.7, u(110) = 1, p = 0.5 \\
\{f_{A,b}, f_{B,r}, f_{B,b}\} & : u(0) = 0, u(100) = 1, u(110) = 1.4, p = 0.5
\end{align*}
\]

Q2. (6.1.19 in MWG) Suppose that an individual has a Bernoulli utility function \( u(x) = -e^{-\alpha x} \) where \( \alpha > 0 \). His (nonstochastic) initial wealth is given by \( w \). There is one riskless asset and there are \( N \) risky assets. The return per unit invested on the riskless asset is \( r \). The returns of the risky assets are independent and normally distributed with means \( \mu = (\mu_1, \ldots, \mu_N) \). Derive the demand function for these \( N + 1 \) assets.

Let \((\sigma_1^2, \ldots, \sigma_N^2)\) be the variances of the risky assets. When the portfolio is \((\alpha_0, \ldots, \alpha_N)\) with \( \sum \alpha_i = 1 \), the expected return is

\[
\begin{align*}
\mathbb{E}[- \exp(-\alpha w (\alpha_0, \ldots, \alpha_N) (r, \ldots, r_N))] &= - \exp(-\alpha w (\alpha_0 r + \sum_{i>0} \alpha_i (\mu_i - \frac{1}{2} \alpha w \alpha_i \sigma_i^2))).
\end{align*}
\]

The expected return is maximized when

\[
\alpha_0 r + \sum_{i>0} \alpha_i (\mu_i - \frac{1}{2} \alpha w \alpha_i \sigma_i^2)
\]

is maximized, and the constraint is

\[
\sum \alpha_i = 1.
\]
We have
\[ \frac{\partial}{\partial \alpha_i} : -r + \mu_i - \alpha w \sigma_i^2 = 0, \]
and
\[ \alpha_i = \frac{\mu_i - r}{\alpha w \sigma_i^2}. \]

Q3. (6.D.3 in MWG) Verify that if a distribution $G(\cdot)$ is an elementary increase in risk from a distribution $F(\cdot)$, then $F(\cdot)$ second-order stochastically dominates $G(\cdot)$.

Let $G(\cdot)$ be an elementary increase from $F(\cdot)$ on the interval $[x', x'']$, and define $I(x) = \int_{x'}^{x} [F(t) - G(t)]dt$. $I(x') = 0$, and by the definition of $G$, $I(x'') = 0$, $I(x) \leq 0$, $\forall x \in [x', x'']$.

\[ \int_{x'}^{x''} u(x)d(F(x) - G(x)) = -\int_{x'}^{x''} u'(x)(F(x) - G(x))dx = \int_{x'}^{x''} u''(x)I(x)dx, \]
and together with $u'' < 0$,

\[ \int_{x'}^{x''} u(x)d(F(x) - G(x)) \geq 0 \]
for any nondecreasing concave function $u$.

Specifically, define $G(\cdot)$ as

\[ G(x) = \begin{cases} 
F(x) & \text{if } x \notin [x', x''] \\
\frac{\int_{x'}^{x''} F(t)dt}{x'' - x'} & \text{if } x \in [x', x''].
\end{cases} \]

This corresponds to $y \sim G, x \sim F, y = x + z$ with

\[ z|x = \begin{cases} 
x' - x & \text{with probability } \frac{x'' - x}{x'' - x'} \\
x'' - x & \text{with probability } \frac{x'' - x'}{x'' - x'}.
\end{cases} \]

Q4. Consider a monopolist who faces a stochastic demand. If he produces $q$ units, he incurs a zero marginal cost and sells the good at price $P(\theta, q)$ where $\theta \in [\theta, \theta]$ is an unknown demand shock where $P$ and $C$ twice differentiable. Assume that the profit function is strictly concave in $q$ for each given $\theta$, and $P(\theta, q) + qP_q(\theta, q)$ is increasing in $\theta$, where $P_q$ is the derivative of $P$ with respect to $\theta$. The monopolist is expected profit maximizer.

(a) Show that there exists a unique optimal production level $q^*$. 

3
(b) Show that if the distribution of $\theta$ changes from $G$ to $F$ where $F$ first-order stochastically dominates $G$, then the optimal production level $q^*$ weakly increases.

(c) Take $P(\theta, q) = \phi(\theta) - \gamma(q)$. Suppose that there are two identical monopolists as above in two independent but identical markets. Find conditions under which the monopolists have a strict incentive to merge and share the profit from each market equally.

(a) Given the zero marginal cost, the monopolist maximizes $\int qP(\theta, q)dF(\theta)$. The profit function is strictly concave in $q$ for every $\theta$

$$\iff \frac{\partial^2}{\partial q^2}(qP(\theta, q)) < 0 \forall \theta, q,$$

and we have

$$\int \frac{\partial^2}{\partial q^2}(qP(\theta, q))dF(\theta) < 0.$$ 

The maximization problem is strictly concave, and there exists a unique optimum $q^*$.

(b) Let $q_G$ and $q_F$ be the optimum for $G$ and $F$, respectively. We have

$$\int (P(\theta, q_G) + q_GP_q(\theta, q_G))dG(\theta) = 0.$$ 

Since $P(\theta, q) + q_GP_q(\theta, q)$ is increasing in $\theta$, when $F$ first-order stochastically dominates $G$,

$$0 = \int (P(\theta, q_F) + q_FP_q(\theta, q_F))dF(\theta)$$

$$= \int (P(\theta, q_G) + q_GP_q(\theta, q_G))dG(\theta)$$

$$\leq \int (P(\theta, q_F) + q_GP_q(\theta, q_F))dF(\theta).$$

By the concavity of the maximization problem, $\int (P(\theta, q) + q_GP_q(\theta, q))dF(\theta)$ is strictly decreasing in $q$, and the optimum for $F$ weakly increases.

(c) If two monopolists share the profit equally, their expected profit is

$$\frac{1}{2} \max_{q_1, q_2} \mathbb{E}[q_1(\phi(\theta_1) - \gamma(q_1)) + q_2(\phi(\theta_2) - \gamma(q_2))]$$

$$= \frac{1}{2} \max_{q_1, q_2}((q_1 + q_2)\mathbb{E}[\phi(\theta)] - q_1\gamma(q_1) - q_2\gamma(q_2)).$$
The profit function is concave in $q$, which implies that $-q\gamma(q)$ is concave in $q$. By Jensen’s inequality, the optimal $q_1$ is the same as $q_2$. Let $q = q_1 + q_2$, then

$$\frac{1}{2}\max_{q_1,q_2}\mathbb{E}[q_1(\phi(\theta_1) - \gamma(q_1)) + q_2(\phi(\theta_2) - \gamma(q_2))]
= \frac{1}{2}\max_q(q\mathbb{E}[\phi(\theta)] - q\gamma(\frac{q}{2}))
= \max(\frac{q}{2}\mathbb{E}[\phi(\theta)] - \frac{q}{2}\gamma(\frac{q}{2})),$$

and the monopolists choose the same quantity as before. They will never have a strict incentive to merge.