14.123 Microeconomic Theory III. 2014

Problem Set 2. Solution.

Anton Tsoy

1. In all counter-examples, I use $s = \frac{1}{3}$, $c = 1$ and suppose that whenever indifferent between $n$ and $n'$, Ann chooses cruder partition. In counter-examples, acts are constant on $[0, \frac{1}{2}]$ and $(\frac{1}{2}, 1]$ and so, it is optimal for Ann to choose only between $n = 0, 1$. I also use notation $E[u(f(s)) : B] = \int_B u(f(s))ds$ and $E_n[u(f(s)) : B] =$

\[ \int_{B \cap (\frac{1}{2}, \frac{1}{2})} u(f(s))ds \]

1.1 Completeness holds. Given $n$ the preferences are complete. Since $u(f(s))$ is bounded, without loss of generality, I can restrict the choice of $n$ to some finite set and so, optimal $n$ exists.

\[ -1 \quad s < \frac{1}{2}, \quad h(s) = \begin{cases} 2 & s < \frac{1}{2}, \\ 1 & s \geq \frac{1}{2}; \end{cases} \quad g(s) = \begin{cases} -2 & s \geq \frac{1}{2}; \end{cases} \]

0. Then $f \sim_s g$ (for $n = 0$, both give expected utility $0$; for $n = 1$, Ann’s expected utility is $\frac{1}{2} < c$ and so, $n = 0$ is optimal)

and $g \sim_s h$ (for $n = 0$, both give expected utility $0$; for $n = 1$, Ann’s expected utility is $\frac{1}{2}0 + \frac{1}{2}2 = 1 = c$ and so, $n = 0$ is optimal),

but $h \succ_s f$ (for $n = 0$, both give expected utility $0$; for $n = 1$, Ann’s expected utility is $\frac{1}{2}1 + \frac{1}{2}2 = \frac{3}{2} > c$ and so, $n = 1$ is optimal).

1.2 Transitivity fails. Consider $f(s) = \begin{cases} -1 & s < \frac{1}{2}, \\ 2 & s \geq \frac{1}{2}; \end{cases}$, $h(s) = \begin{cases} 2 & s < \frac{1}{2}, \\ -2 & s \geq \frac{1}{2}; \end{cases}$

\[ g(s) = \begin{cases} 1 & s \geq \frac{1}{2}; \end{cases} \]

0. Then $f \sim_s g$ (for $n = 0$, both give expected utility $0$; for $n = 1$, Ann’s expected utility is $\frac{1}{2} < c$ and so, $n = 0$ is optimal)

and $g \sim_s h$ (for $n = 0$, both give expected utility $0$; for $n = 1$, Ann’s expected utility is $\frac{1}{2}0 + \frac{1}{2}2 = 1 = c$ and so, $n = 0$ is optimal),

but $h \succ_s f$ (for $n = 0$, both give expected utility $0$; for $n = 1$, Ann’s expected utility is $\frac{1}{2}1 + \frac{1}{2}2 = \frac{3}{2} > c$ and so, $n = 1$ is optimal).

1.3 P2 holds. Let $f, f', g, g'$ be defined as in P2 in the lecture notes for some $B \subset S$.

Let $n$ and $n'$ be optimal levels of contemplation for comparison of $f$ and $g$, and $f'$ and $g'$, respectively. Observe that $n = n'$, since

\[ E_n[u(f(s)) - E_n[u(g(s))] = E_n[u(f(s)) : B] - E_n[u(g(s)) : B] = \]

\[ = E_n[u(f'(s)) : B] - E_n[u(g'(s)) : B] = E_n[u(f'(s)) - E_n[u(g'(s))]. \]

Then

\[ f \succeq_s g \iff E_n[u(f(s)) \geq E_n[u(g(s))] \iff E_n[u(f(s)) : B] \geq E_n[u(g(s)) : B] \iff \]

\[ \iff E_n[u(f'(s)) : B] \geq E_n[u(g'(s)) : B] \iff E_n[u(f'(s)) \geq E_n[u(g'(s)) \iff f' \succeq_s g'. \]

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1.4 P3 holds. Consider \( x, x' \) and \( f = x^h_{|B}, f' = x'^h_{|B} \) for some act \( h \in F \) and \( B \subset S \). Observe that \( n = 0 \) is optimal for comparison of \( f \) and \( f' \), as \( f \) is (weakly) better than \( f' \) state by state. Therefore, \( f \succ f' \iff \mathbb{E}u(f(s)) > \mathbb{E}u(f'(s)) \iff x \succ x' \), since \( B \) is non-null.

1.5 P4 fails. Consider \( A = [0, \frac{1}{2}], B = (\frac{1}{2}, 1], \) and \( x = 1, x' = -1, y = 2, y' = -2. \) Then \( f_A \sim_s f_B \) (for \( n = 0 \), both give expected utility 0; for \( n = 1 \), Ann’s expected utility is 1, but she needs to incur contemplation costs \( c = 1 \) and so, \( n = 0 \) is optimal), but \( g_A \succ_s g_B \) (as before, for \( n = 0 \), both give expected utility 0; for \( n = 1 \) Ann’s expected utility is 2 which after subtracting contemplation costs gives payoff 1 and so, \( n = 1 \) is optimal).

1.6 P5 holds by the assumption that \( Z \) contains at least two elements.

2. I am looking for a concave utility function \( u \in \mathcal{U} \) that satisfies

\[
\frac{1}{2}u(\omega_0 + G) + \frac{1}{2}u(\omega_0 - L) = u(\omega_0) \quad \text{and} \quad \frac{6}{5}u(\omega_0 + 100) + \frac{4}{5}u(\omega_0 - 100) = u(\omega_0)
\]

with the smallest reward \( G \). To find such \( G \), I need to find \( u \in \mathcal{U} \) such that the utility gain from \( G \) is as big as possible, and the utility loss from \( L \) is as small as possible. Given the restriction to concave functions, it’s clear that the optimal \( u \) should be linear on intervals where it is not specified by constraints on \( u \) and should match the derivatives at the boundaries of intervals.

2.1 Here, \( u \) is only specified at three points \((\omega_0, u(\omega_0)), (\omega_0 + 1, u(\omega_0 + 1)), (\omega_0 - 1, u(\omega_0 - 1))\), so we do linear extrapolation on the rest of the domain. Therefore,

\[
\frac{u(\omega_0 + G) - u(\omega_0)}{u(\omega_0) - u(\omega_0 - L)} = \frac{2G}{3L},
\]

and at the same time

\[
\frac{u(\omega_0 + G) - u(\omega_0)}{u(\omega_0) - u(\omega_0 - L)} = 1,
\]

so \( G = 150000. \)

2.2 Here, \( u \) is specified only on \([\omega_0 - 100, \omega_0 + 100]\). From \( .6e^{-\alpha(\omega_0+1)} + .4e^{-\alpha(\omega_0-1)} = e^{-\alpha \omega_0} \), find \( \alpha = \ln 1.5 \) and so, \( u'(\omega_0 + 100) = \ln 1.5(1.5)^{-(\omega_0 + 100)} \) and \( u'(\omega_0 - 100) = \ln 1.5(1.5)^{-(\omega_0 - 100)} \). By the linearity of \( u \) outside \([\omega_0 - 100, \omega_0 + 100]\),

\[
u(\omega_0 + 100 + (G - 100)) = u(\omega_0 + 100) + u'(\omega_0 + 100)(G - 100),
\]

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\[ u(\omega_0 - 100 - (L - 100)) = u(\omega_0 - 100) - u'(\omega_0 - 100)(L - 100). \]

Now using the second indifference condition, we get

\[ G = 100 + \frac{-2 + 1.5^{100} + 1.5^{-100} + \ln 1.5(1.5)^{100}(L - 100)}{\ln 1.5(1.5)^{-100}}. \]

2.3 From the previous part \( \alpha = \ln 1.5 \) and so, \( \left(\frac{2}{3}\right)^G + \left(\frac{2}{3}\right)^{-L} = 2 \) which is not possible.

2.4 Let \( x = \omega_0^{-1} \). By CRRA specification,

\[ .6(1 + x)^{1-\rho} + .4(1 - x)^{1-\rho} = 1 = .5(1 + Gx)^{1-\rho} + .5(1 - Lx)^{1-\rho}. \]

By \( \omega_0 \geq L \gg 0 \), we have \( x \ll 0 \) and so, we could take the Taylor expansion of the first equation to get

\[ .6(1 - \rho)x + .6(1 - \rho)\rho \frac{x^2}{2} = .4(1 - \rho)x - .4(1 - \rho)\rho \frac{x^2}{2} + o(x^2) \]

or \( \rho x = .4 + o(x^2) \) and so, \( \rho \gg 1 \). Now \( (1 - Lx)^{1-\rho} \geq \exp(Lx(\rho - 1)) \approx \exp(.4L\frac{\rho - 1}{\rho}) \gg 2 \) and so, it is impossible to find appropriate \( G \).