14.124 Problem Set 2

Question 1

(a) The program is the following:
\[
\begin{align*}
\text{min } & \sum_i p_H(i)s_i; \\
\text{s.t. } & \sum_i [p_H(i) - p_L(i)]s_i \geq c(H); \\
& s_i \geq 0, \text{ for all } i.
\end{align*}
\]
The first constraint is the agent’s IC constraint, and the second is the limited liability constraint.

(b) Let \(\mu\) and \(\nu_i\) be the Lagrange multipliers of the two constraints, respectively. Then the first-order conditions can be written as follows:
\[
1 - \frac{p_H(i) - p_L(i)}{p_H(i)}\mu - \nu_i = 0, \text{ for all } i.
\]
Notice that all multipliers are non-negative and a multiplier is zero only if its corresponding constraint is not binding.

Suppose that \(\mu = 0\). Then \(\nu_i = 1\) for all \(i\) which implies that \(s_i = 0\) for all \(i\), which cannot satisfy the agent’s IC constraint. Therefore, \(\mu > 0\). The strict MLRP implies that \((p_H(i) - p_L(i))\mu/p_H(i)\) is strictly increasing in \(i\), which further implies that \(\nu_i\) is strictly decreasing in \(i\). Therefore, \(\nu_1 > 0\) and \(s_i = 0\) for \(i = 1, 2, ..., n - 1\). Finally, \(s_n > 0\) since otherwise the agent’s IC constraint cannot be satisfied.
Question 2

(a) Let \( p_{NF} \) be the probability that there is no fire, \( p_{2e} \) the probability that damage is 2000 when Adam exerts effort and \( p_{2ne} \) the probability that damage is 2000 when Adam does not exert effort. Let \( x_1, x_2 \) and \( x_3 \) be Adam’s consumption in these three cases. Here is the program:

\[
\begin{align*}
\max_{x_1, x_2, x_3} & \quad p_{NF} u(x_1) + (1 - p_{NF})(1 - p_{2e})u(x_2) + (1 - p_{NF})p_{2e}u(x_3) \\
\text{s.t.} & \quad (1 - p_{NF})(p_{2ne} - p_{2e})(u(x_2) - u(x_3)) \geq c; \\
& \quad p_{NF}(-x_1) + (1 - p_{NF})(1 - p_{2e})(-1000 - x_2) + (1 - p_{NF})p_{2e}(-2000 - x_3) \geq -W,
\end{align*}
\]

where \( W \) is Adam’s initial wealth.

(b) If Adam’s IC constraint is not binding, \( x_2 = x_3 \) at optimality and the IC constraint is violated. If the company’s IR constraint is not binding, then all \( x \)'s are infinite. Therefore, both constraints are binding. Let \( \lambda > 0 \) and \( \mu > 0 \) be Lagrange multipliers of the two constraints. Then the FOCs are

\[
\begin{align*}
u'(x_1) - \mu &= 0; \\
\frac{u'(x_2) + p_{2ne} - p_{2e}}{1 - p_{2e}} \lambda - \mu &= 0; \\
\frac{u'(x_3) - p_{2ne} - p_{2e}}{p_{2e}} \lambda - \mu &= 0.
\end{align*}
\]

Therefore, \( u'(x_2) < u'(x_1) < u'(x_3) \). Since \( u \) is strictly concave, \( u' \) is strictly decreasing, so \( x_3 < x_1 < x_2 \). The same ordering is true for the \( S \)'s. Here is the intuition: we want to encourage Adam to exert the effort, so we need \( x_2 > x_3 \) to give him incentive. On the other hand, we want to insure him against the risk of fire, so we want to choose \( x_1 \) in between.

(c) Since the damage of 2000 does not occur in reality, we can make Adam’s payoff as low as possible in that case. Then we set \( S_1 = S_2 \) and check if the IC constraint is satisfied. If for some reason (such as limited liability) the IC constraint is not satisfied, then \( S_1 < S_2 \).
Question 3

(a) The agent maximizes

\[ \hat{w}(e, s; \alpha, \beta) = \beta + \alpha e - r \frac{\alpha^2}{2} \sigma^2 - \frac{1}{2}(e + s)^2 + s. \]

For \( \alpha = 0.3, e = 0 \) and \( s = 1 \).

(b) In this case, the agent chooses effort \( \alpha \), and the principal maximizes

\[ (1 - \alpha)e - \beta = \alpha - \frac{r}{2} \frac{\alpha^2}{2} \sigma^2 - \frac{1}{2} \alpha^2, \]

where \( \underline{u} \) is the agent’s reserved utility. Therefore, the optimal \( \alpha \) is \( 1/(r \sigma^2 + 1) = 1/9 \).

(c) In this case, \( \underline{u} = \max_s s - \frac{1}{2} s^2 = \frac{1}{2} \), so

\[ \beta = \underline{u} + r \frac{\alpha^2}{2} \sigma^2 - \alpha e + \frac{1}{2} e^2 = \frac{44}{81}. \]
Question 4

(a) If \( e \) can be contracted on, the principal can write a contract to maximize total surplus, eliminate risk to the agent, and place the agent exactly at the minimum utility by choosing \( e \) and paying only if that level of \( e \) is observed. Note that with a risk averse agent and risk neutral principal, optimal risk sharing requires the principal to bear all risk and pay the agent a fixed amount. The principal maximizes

\[
\max_{e,s} \mathbb{E}[x|e] - s \quad \text{s.t.} \quad u(s) - c(e) \geq u
\]

The FOCs give

\[
1 = \lambda u'(s^*) \quad \Rightarrow \quad u'(e^*) = c'(s^*)
\]

This condition together with the participation constraint gives a unique first-best effort \( e^* \) and associated payment \( s^* \).

(b) For simplicity, I assume \( c(e) \geq 0, \forall e \); the problem can be solved without this but it is slightly more tedious. Define \( \underline{s} \) such that \( u(s) = \min\{u_e, u(s^*) - u'(s^*)\} \); by construction \( \underline{s} < s^* \). Consider the contract

\[
s(x) = \begin{cases} 
  s^* & \text{if } x \in [e^*, e^* + 1] \\
  \underline{s} & \text{otherwise}
\end{cases}
\]

When the agent chooses \( e = e^* \), then the agent’s utility is \( u \); this is the first-best efficient solution and satisfies IR by construction. All that remains is to verify IC.

If the agent chooses \( e > e^* \), then she increases her disutility of effort and lowers her expected payout by placing positive probability on outcomes that pay \( \underline{s} < s^* \), therefore she will not choose \( e > e^* \). When \( e \leq e^* - 1 \), the agent’s expected utility is at most \( \underline{u} - c(e) \leq u \) so the agent will not prefer that either. For \( e = e^* - \delta, \delta \in (0, 1) \), the agent’s expected utility is

\[
\mathbb{E}[U_A|e] = \mathbb{E}[s(x)|e] - c(e)
\]

\[
\leq (1 - \delta)u(s^*) + \delta u(\underline{s}) - (c(e^*) - \delta c'(e^*))
\]

\[
= \{u(s^*) - c(e^*)\} - \delta(u(s^*) - u(\underline{s}) - c'(e^*))
\]

\[
\leq \mathbb{E}[U_A|e^*] - \delta[u'(s^*) - c'(e^*)]
\]

\[
= \mathbb{E}[U_A|e^*]
\]

where the first inequality follows from the convexity of \( c(e) \). Thus the agent will not prefer \( e \) in that region, meaning IC is satisfied and the first-best is achieved.