Question 1

(a) Let $t_i$ be the total tax payment by the monopoly of type $i$. The program is as follows:

$$\max_{t_i, x_i} \sum_i p_i t_i;$$

s.t. $x_i - c_i x_i - t_i \geq 0$, for all $i$;

$$x_i - c_i x_i - t_i \geq x_j - c_j x_j - t_j, \text{ for all } i, j.$$

(b) The firm’s profit has strictly decreasing differences in $c$ and $x$, so any implementable $x_i$ must be weakly decreasing in $i$. To see this, simply add up the IC constraints that types $i$ and $j$ do not imitate each other, and the $t_i$ and $t_j$ cancel out and we obtain that

$$-c_j x_j - c_i x_i \geq -c_j x_i - c_i x_j.$$

It can be rewritten as $(c_i - c_j)(x_i - x_j) \leq 0$. Therefore, if $i > j$ (so that $c_i > c_j$), then $x_i \leq x_j$.

(c) Since the firm’s profit is decreasing in $i$, the IR constraint is only binding for type $n$, which means that $t_n = (1 - c_n)x_n$. At optimality type $i$ is indifferent between reporting $i$ and reporting $(i + 1)$, so $t_i - t_{i+1} = (1-c_i)x_i - (1-c_i)x_{i+1}$. (Convince yourself this fact if you did not come to recitation.) Therefore,

$$t_i = (1 - c_i)x_i - \sum_{j=i+1}^n (c_j - c_{j-1})x_j. \quad (1)$$
The summation is type $i$’s profit. Therefore, the regulator’s maximum payoff under a production plan $(x_i)$ is

$$\sum_i p_i \left[ (1 - c_i)x_i - \sum_{j=i+1}^n (c_j - c_{j-1})x_j \right] = \sum_i \left[ p_i(1 - c_i) - \sum_{k=1}^{i-1} p_k(c_i - c_{i-1}) \right] x_i.$$  

The constraint is that $0 \leq x_n \leq x_{n-1} \leq \ldots \leq x_1 \leq 1$. Since the objective function is linear in $x_i$, each $x_i$ reaches its upper bound ($x_{i-1}$ if $i > 1$ or 1 if $i = 1$) or lower bound ($x_{i+1}$ if $i < n$ and 0 if $i = n$). Therefore, there exists a $k$ such that $x_i = 1$ when $i \geq k$ and $x_i = 1$ if $i > k$. By Eq. (1), $t_i = 0$ for $i > k$ and for $i \leq k$,

$$t_i = 1 - c_i - \sum_{j=i+1}^k (c_j - c_{j-1}) = 1 - c_k.$$  

**Question 2**

Notice that $c(q, \beta)$ has strictly increasing differences in $q$ and $\beta$, so every non-increasing $q(\beta)$ can be implemented. Indeed $q(\beta) = 1/\beta^2$ is decreasing in $\beta$. The envelope theorem implies that the transfer schedule $t(\beta)$ that implements $q(\beta)$ is unique, and is given by

$$t(\beta) = c(q(\beta), \beta) + \pi(\beta_0) - \int_{\beta_0}^{\beta} c_2(q(\tilde{\beta}), \tilde{\beta}) d\tilde{\beta},$$

where $\beta_0$ is a type, $\pi(\beta_0)$ is a constant, and $c_2$ is the partial derivative of $c$ with respect to its second argument. Substituting in $q(\beta) = 1/\beta^2$, we obtain that

$$t(\beta) = \frac{1}{\beta} + \pi(\beta_0) - \int_{\beta_0}^{\beta} \frac{1}{\beta^2} d\tilde{\beta} = \frac{2}{\beta} + \pi(\beta_0) - \frac{1}{\beta_0}.$$  

Let $A = \pi(\beta_0) - \frac{1}{\beta_0}$. Notice that $\beta = q^{-1/2}$, so

$$p(q) = t(q^{-1/2}) = 2\sqrt{q} + A.$$
Question 3

(a) Since utilities are transferable, the efficient trading rule maximizes the total surplus \((v - c)x\), which means that \(x = 1\) if and only if \(v \geq c\).

(b) Let \(x(m_S, m_B)\) be the trading rule, and \(t_S(m_S, m_B)\) and \(t_B(m_S, m_B)\) be the transfer rule. Then in a direct mechanism that implements efficient trade, \(x(m_S, m_B) = 1\) if \(m_B \geq m_S\) and 0 otherwise. The seller’s payoff under this mechanism is

\[
t_S(m_S, m_B) - cx(m_S, m_B).
\]

It is required that \(m_S = c\) is optimal for all \(m_B\). Since the seller can choose the \(m_S\) that maximizes \(t_S(m_S, m_B)\) under the same physical allocation \(x(m_S, m_B)\), there exist two functions \(t_{S1}(m_B)\) and \(t_{S0}(m_B)\) such that \(t_S(m_S, m_B) = t_{Sx(m_S,m_B)}(m_B)\). In other words, the payment that the seller receives only depends on the buyer’s message and whether trade occurs. IC constraints imply that

\[
t_{S1}(m_B) - c \geq t_{S0}(m_B), \text{ if } m_B \geq c;
\]

\[
t_{S0}(m_B) \geq t_{S1}(m_B) - c, \text{ if } m_B < c.
\]

Therefore, \(t_{S1}(m_B) - t_{S0}(m_B) = m_B\). Similarly, there exists a function \(t_{B0}\) such that \(t_B(m_S, m_B) = t_{B0}(m_S)\) when \(m_S > m_B\) and \(t_B(m_S, m_B) = t_{B0}(m_S) + m_S\) when \(m_S \leq m_B\).

(c) This is obvious from the previous part as the requirement forces both \(t_{S0}\) and \(t_{B0}\) to be zero.

(d) Notice that \(t_B(m_S, m_B) - t_S(m_S, m_B) = \min\{m_S - m_B, 0\}\), so the budget breaks whenever \(c < v\).
Question 4

(a) Under this mechanism, a buyer’s payoff is \((v_b - p)m_b(\hat{v}_b)\) if she reports value \(\hat{v}_b\). Clearly reporting the true \(v_b\) is optimal. Since the buyer’s payoff is always non-negative, she always participates. Similarly, the IC and IR constraints are satisfied for the seller.

(b) The total transfer from the mechanism designer is 
\[-[1 - F_b(p)]p + F_s(p)p = [F_s(p) + F_b(p) - 1]p.\]

Next \(F_s(p) + F_b(p) = 0\) when \(p = -\infty\) and \(F_s(p) + F_b(p) = 2\) when \(p = \infty\), so there exists a \(p^*\) such that

\[F_s(p^*) + F_b(p^*) = 1,\]

which means that the budget is balanced when the price is \(p^*\). The above condition also means that the mass of sellers who trade \((F_s(p^*))\) and the mass of buyers who trade \((1 - F_b(p^*))\) are equal, so the mechanism is feasible.

The efficient trade scheme must satisfy five conditions:

- It is feasible: the mass of buyers who trade equals the mass of sellers who trade;

- If a seller of value \(v_s\) trades, all sellers of lower values trade;

- If a buyer of value \(v_b\) trades, all buyers of higher values trade;

- If a buyer with value \(v_b\) trades and a seller with value \(v_s\) trades, then \(v_b \geq v_s\);

- If a buyer with value \(v_b\) does not trade and a seller with value \(v_s\) does not trade, then \(v_b \leq v_s\).

The second and the third requirements mean that the efficient scheme is characterized by two thresholds \(\bar{v}_s\) and \(v_b\) such that a seller trades if and only if his value is below \(\bar{v}_s\) and a buyer trades if and only if her value is higher than \(v_b\). The fourth and fifth requirements mean that \(\bar{v}_s = v_b\). The first requirement implies that 
\[1 - F_b(v_b) = F_s(\bar{v}_s).\]
Clearly, this condition means that \(\bar{v}_s = v_b = p^*\).
(c) This mechanism will not be feasible (i.e. sometimes a buyer wants to buy but the seller does not want to sell or the other way around) when there is only one buyer and only one seller.