Adverse Selection

- We will solve a procurement problem using a screening mechanism.
- Idea: buyer wants to buy from seller, but doesn’t know seller’s cost.
Two players, buyer $B$ and seller $S$

- $v(x)$: value of $x$ units to $B$
- $c(x, \theta)$: cost of producing $x$ by $S$ depending on his type $\theta$
- Payoffs: $u_B(x, t) = v(x) - t$, $u_S(x, t, \theta) = t - c(x, \theta)$, where $t$ is payment from $B$ to $S$
- Assumptions: $v' > 0$, $v'' \leq 0$, $v(0) = 0$
- $c_{x\theta} < 0$ (higher types have lower marginal cost), $c(0, \theta) = 0$ $\forall \theta$, $c_x > 0$ (positive MC)
- $B$ designs $t(x)$, a nonlinear price schedule specifying a payoff for each quantity.
- Given $t(x)$, under some conditions, a seller of type $\theta$ will choose a quantity $x(\theta)$ such that marginal cost equals marginal payoff from one more unit: $c_x(x(\theta), \theta) = t'(x)$.
- Note: no matter how $B$ designs $t(x)$, lower cost sellers always produce more.
- Easiest to prove using increasing differences.
Note: if there are $k$ (finitely many) types, I only need $t$ to specify payoffs for $k$ product amounts to implement any outcome.

In equilibrium, given some $t$, types $\theta_1, \ldots, \theta_k$ choose amounts $x_1, \ldots, x_k$ respectively, so we can design $t_2$ that pays $t_2(x_i) = t(x_i)$ and $t_2(x) = 0$ otherwise: $t_2$ implements the same outcome.

So in the 2 type case, we only need to choose two pairs $(x_1, t_1)$, $(x_2, t_2)$ such that type 1 wants to choose $x_1$ and 2 chooses $x_2$.

Another of those reformulations that are mathematically equivalent but make the problem more tractable.
Types $\theta_1, \theta_2$: $Pr(\theta_1) = p$, $Pr(\theta_2) = 1 - p$

Cost functions $c_1(x), c_2(x)$

$B$ chooses $\{(x_1, t_1), (x_2, t_2)\}$ to solve:

$$\max p(v(x_1) - t_1) + (1 - p)(v(x_2) - t_2)$$

s.t.

$$t_1 - c_1(x_1) \geq t_2 - c_1(x_2) \quad (IC1)$$
$$t_2 - c_2(x_2) \geq t_1 - c_2(x_1) \quad (IC2)$$
$$t_1 - c_1(x_1) \geq 0 \quad (IR1)$$
$$t_2 - c_2(x_2) \geq 0 \quad (IR2)$$
Note: one weird thing about this setup is both types have the same outside option

Rarely true in reality

Note 2: the IC conditions are analogous to requiring tangency in the continuous case

But here “tangency” is not meaningful because there are only 2 options

Note 3: there may be solutions where we decide to exclude the low type altogether and just offer one pair \((x_2, t_2)\), but we will come back to that later
General intuition: in the optimal solution, 1’s IR constraint will bind but not his IC, and 2’s IC constraint will bind but not his IR.

Why?

Since 2 has lower cost for any \( x \), if 1’s IR constraint holds, 2’s must hold with slack (could at worst produce \( x_1 \) and make positive profit)

\[
t_2 - c_2(x_2) \geq t_1 - c_2(x_1) > t_1 - c_1(x_1) \geq 0
\]

Hence 2’s IR never binds.

If 1’s IR did not bind, \( B \) could lower both \( t_1 \) and \( t_2 \) by the same amount and make more money.

Hence 1’s IR binds.
Since 2 has lower marginal cost and \( x_2 > x_1 \), it can’t be that IC1 and IC2 both bind.

If IC1 binds, 1 is indifferent between \( x_1 \) and \( x_2 \), but then 2 strictly prefers \( x_2 \), hence IC2 does not bind.

If IC2 binds, 2 is indifferent, hence 1 strictly prefers \( x_1 \), hence IC1 does not bind.

Whenever IC2 does not bind, \( B \) can improve by lowering \( t_2 \) a little:
- 2 still chooses \( x_2 \)
- 1 chooses \( x_1 \) even more strongly and his IR is unaffected
- 2’s IR is not violated if change is small enough since it wasn’t binding

Hence in optimal solution IC2 must bind, hence IC1 does not bind.
So $B$ first chooses a point on 1’s zero-profit curve, i.e., $B$ chooses $x_1$ and $t_1 = c_1(x_1)$

And then moves up 2’s cost curve up to some point, i.e., $B$ chooses $x_2$ and $t_2 = t_1 - c_2(x_1) + c_2(x_2)$

So how to choose $x_1, x_2$?

$x_2$ can just be picked as first-best!

Whatever $x_1$ is, changing $x_2$ does not affect 1’s incentives, just how much 2 produces and how much $B$ pays 2

So can just choose $x_2$ such that $c_2'(x_2) = v'(x_2)$ (first-best)
What about $x_1$?

Picking the first-best $x_1$ is not good: the more I increase $x_1$, not only do I have to pay 1 more, but also have to pay 2 more at the same $x_2$ to satisfy his IC.

For the same reason, $x_1$ higher than FB is also bad, and optimal $x_1$ is below FB.

The FOC is: $\rho = c'_1(x_1) - (1 - \rho)c'_2(x_1) > pc'_1(x_1)$.
If $p < c'_1(x_1) - (1 - p)c'_2(x_1)$ even for small $x_1$, then may want to choose $x_1 = 0$ (price 1 out of the market).

- $p$ does not affect $x_2$, but it affects $x_1$
- The lower $p$ is, the lower $x_1$ is
Main tension in this model is between desire to produce at the efficient level (choose $x_1$, $x_2$ equal to FB levels) and $B$’s desire to limit type 2’s rent

- Have to screw over type 1 to reduce type 2’s temptation
- If $p$ is low, lowering $x_1$ has low efficiency cost (low type is unlikely anyway) but big rent reduction ($B$ pays less to the likely high type)
- Vice versa for high $p$
How to derive the FOC: the problem is reduced to

\[
\max p(v(x_1) - t_1) + (1 - p)(v(x_2) - t_2)
\]
\[
\text{s.t. } t_2 - c_2(x_2) = t_1 - c_2(x_1) \quad \text{(IC2)}
\]
\[
\quad t_1 - c_1(x_1) = 0 \quad \text{(IR1)}
\]

Or equivalently

\[
\max p(v(x_1) - c_1(x_1)) + (1 - p)(v(x_2) - c_2(x_2) - c_1(x_1) + c_2(x_1))
\]
\[
\Longrightarrow p(v'(x_1) - c_1'(x_1)) + (1 - p)(-c_1'(x_1) + c_2'(x_1)) = 0
\]
\[
(1 - p)(v'(x_2) - c_2'(x_2)) = 0
\]
Reminder: we were solving the screening problem, which we had reduced to:

$$\max p (v(x_1) - c_1(x_1)) + (1 - p) (v(x_2) - c_2(x_2) - c_1(x_1) + c_2(x_1))$$

(s.t. $x_2 \geq x_1$)

But the condition $x_2 \geq x_1$ does not bind so we can ignore it.

We get the FOCs:

$$p (v'(x_1) - c_1'(x_1)) + (1 - p) (-c_1'(x_1) + c_2'(x_1)) = 0$$

$$(1 - p) (v'(x_2) - c_2'(x_2)) = 0$$
From the second FOC, $v'(x_2) = c'_2(x_2)$, so $x_2 = x_2^{FB}$, the first-best value

Here “first-best” means the value that maximizes the total surplus of the principal and agent

And also the value that would result from the optimal contract *if the agent were known to be type 2*

From the first FOC,

$$v'(x_1) - c'_1(x_1) = \frac{1-p}{p}(c'_1(x_1) - c'_2(x_1)) > 0,$$

so $x_1^* < x_1^{FB}$
Hence the principal designs the menu \{ (x_1, t_1), (x_2, t_2) \} so that type 1 underproduces in equilibrium.

Again, this is to make it cheaper to prevent type 2’s temptation to fake being type 1.

In particular, \( x_2^* = x_2^{FB} > x_1^{FB} > x_1^* \).

If \( p \) is high, there is less distortion in \( x_1 \) so \( x_1^* \) goes up.

If \( p \) is low enough, can go all the way to \( x_1 = 0 \) (type 1 is shut out of the market).
An alternative way to think about the problem of choosing $x_1$

We can define

$$\tilde{c}(x_1) \equiv c_1(x_1) + \frac{1 - p}{p} \Delta c(x_1)$$

Then the choice of $x_1$ made in the screening mechanism is actually the FB choice, for a hypothetical agent that had this (higher) cost function.

The cost function captures both the real cost of 1 producing more $x$, and the cost of having to pay type 2 more as a result of increasing $x_1$. 
Suppose I have types $\theta_1, \ldots, \theta_m$

Cost functions $c_1, \ldots, c_m$ such that $c'_i(x) > c'_j(x)$ for all $i < j$ and any $x$ (higher types have lower marginal cost)

Probabilities $p_1, \ldots, p_m$ adding up to 1

How to design the mechanism?
As before, we need to define at most \( m \) points: \((t_1, x_1), \ldots, (t_m, x_m)\)

Could be fewer if I want to shut out some types, but not more (can just drop options from the contract which no one picks in equilibrium anyway)

Now there are \( m \) IR constraints: \( \text{IR}_1, \ldots, \text{IR}_m \)

How many IC constraints? For each type \( k \), need one IC constraint for each \( i \neq k \), saying \( k \) prefers picking \((t_k, x_k)\) to \((t_i, x_i)\)

So \( k(k-1) \) IC constraints: \( \text{IC}_{k1}, \ldots, \text{IC}_{k(k-1)}, \text{IC}_{k(k+1)}, \ldots, \text{IC}_{kn} \)
Which ones bind?

We can show (with similar arguments to the 2-state case) that:
- Only IR\(_1\) binds (higher types have lower cost so necessarily positive profits)
- Only IC\(_{k(k-1)}\) binds for each \(k = 2, \ldots, n\)

Lowest type who is not priced out is left indifferent about entering
Each type is indifferent about not mimicking the next type with higher cost
(But strictly does not want to mimic others)
This gives the right amount of conditions: *given* some values of $x_1, \ldots, x_m$, the conditions uniquely pin down $t_1, \ldots, t_m$

- From IR$_1$, we know $t_1 = c_1(x_1)$: pins down $t_1$
- From IC$_{21}$, we know that $t_2 - c_2(x_2) = t_1 - c_2(x_1)$: pins down $t_2$
- And so on
- Finding the optimal $x_1, \ldots, x_m$ still requires solving for some FOCs

(Side note: choosing $t_i$’s with this algorithm allows us to implement *any* sequence $x_1, \ldots, x_m$ we want, as long as it’s non-decreasing, but some are better for the principal than others)
\[ x_m^* = x_m^{FB}, \text{ but for } i < m \text{ we will have } x_i^* < x_i^{FB} \]

- As before, increasing \( x \) for low types forces principal to pay all higher types more (by the same amount)
- Hence distortion is worst for the lowest \( i \)'s (highest cost types)
Continuous Case

- Suppose now we have a continuum of types $\theta \in [0, 1]$
- $\theta$ distributed according to a continuous cdf $F$, with density $f$
- (Could deal with atoms in distribution; holes in the support are more annoying)
- Suppose $c_{x\theta} < 0$, $c(0, \theta) = 0$ for all $\theta$, and (hence) $c_\theta < 0$
- Higher types have lower marginal cost, hence lower cost
Now principal solves:

\[
\max_{x(\cdot), t(\cdot)} \int_0^1 (x(\theta) - t(\theta)) \, dF(\theta)
\]

s.t. \[ t(\theta) - c(x(\theta), \theta) \geq t(\theta') - c(x(\theta'), \theta) \quad \forall \theta, \theta' \quad (IC_{\theta, \theta'}) \]

\[ t(\theta) - c(x(\theta), \theta) \geq 0 \quad \forall \theta \quad (IR_\theta) \]
- Define $\Pi(\tilde{\theta}, \theta) \equiv t(\tilde{\theta}) - c(x(\tilde{\theta}), \theta))$
- This is the profit $\theta$ gets from pretending to be $\tilde{\theta}$
- Define $V(\theta) \equiv \Pi(\theta, \theta)$
- This is type $\theta$’s equilibrium payoff
- Then the IC conditions can be rewritten as $V(\theta) \geq \Pi(\tilde{\theta}, \theta)$ for all $\theta, \tilde{\theta}$
What do our conditions imply about \( V(\theta) \)?

Since it's the value function of an optimization problem, we can use the envelope theorem:

\[
V'(\theta) = \frac{d\Pi(\theta, \theta)}{d\theta} = \frac{\partial \Pi(\theta_1, \theta_2)}{\partial \theta_2} \bigg|_{(\theta, \theta)} = -c_\theta(x(\theta), \theta) > 0
\]
Note: $V(\theta)$ a priori doesn’t have to be differentiable, as it is endogenous: the principal could pick a non-smooth $x$ or $t$

But we know $c_\theta$ is well-defined by assumption

There are versions of the envelope theorem for non-differentiable functions, which guarantee we can use it without knowing ex ante that $V$ is differentiable

But too complicated for this class, so just assume functions are differentiable
Now we can integrate $V'(\theta)$:

$$V(\theta) = \Pi(0, 0) - \theta_0 c_\theta(x(\tilde{\theta}), \tilde{\theta}) d\tilde{\theta}$$

Since $V(\theta) = t(\theta) - c(x(\theta), \theta)$,

$$t(\theta) = \Pi(0, 0) + c(x(\theta), \theta) - \theta_0 c_\theta(x(\tilde{\theta}), \tilde{\theta}) d\tilde{\theta}$$
This has a similar flavor to the finite types case: given some $x(\theta)$, we can pin down $t(\theta)$

But it is not logically equivalent!

In the finite case, given $x_1, \ldots, x_m$, there were many $t_1, \ldots, t_m$ that could be paired with them that would implement production $x_1$ for $\theta_1$, ..., $x_m$ for $\theta_m$

The uniqueness of the $t_i$ followed from making some IR and IC conditions bind, to achieve optimality for the principal

(You could design other $t_i$ schedules such that no IR or ICs would bind, and which would also implement the same $x_i$’s, but they would give some agent types free money)
On the other hand, in the continuous case, the conditions which uniquely pin down \( V(\theta) \) and \( t(\theta) \) (up to \( \Pi(0, 0) \)) follow exclusively from the assumption that picking the schedule \( x(\theta) \) is optimal (i.e., incentive compatible) for the agent.

We have not yet exploited in any way the assumption that we’re trying to achieve optimality for the principal!

The only way optimality for the principal will show up, in terms of conditions on \( t \), is that we should set \( \Pi(0, 0) = 0 \) (no free money for lowest type).

But we still have to find the optimal schedule \( x(\theta) \).
The problem

\[
\max_{x(\cdot), t(\cdot)} \int_0^1 (x(\theta) - t(\theta)) \, dF(\theta)
\]

now becomes

\[
\max_{x(\cdot)} \int_0^1 \left( x(\theta) - c(x(\theta), \theta) + \int_0^\theta c_\theta(x(\tilde{\theta}), \tilde{\theta}) \, dF(\theta) \right) \, dF(\theta)
\]

Subject only to the condition that \(x(\theta)\) is non-decreasing
Changing the order of integration, we can rewrite this as

$$\max_{x(\cdot)} \int_0^1 (x(\theta) - \tilde{c}(x(\theta), \theta)) f(\theta) d\theta$$

where

$$\tilde{c}(x(\theta), \theta) \equiv c(x, \theta) - c_{\theta}(x, \theta) \frac{1 - F(\theta)}{f(\theta)}$$
Deriving with respect to each $x(\theta)$, we get the FOC:

$$c_x(x, \theta) - c_{x\theta}(x, \theta) \frac{1 - F(\theta)}{f(\theta)} = 1 \quad \forall \theta$$

This gives us an equation in $x(\theta)$ which generally pins down $x(\theta)$

As before, the solution satisfies that $x^*(\theta) < x^{FB}(\theta)$ for $\theta < 1$, and $x^*(1) = x^{FB}(1)$
One question left: is the solution $x^*(\theta)$ pinned down by this condition necessarily non-decreasing?

Not always!

It turns out that, when the solution to this system of FOCs is non-monotonic, you can find the “real” solution by smoothing out the decreasing parts.

Surprisingly, this does not affect the optimal value of $x(\theta)$ outside of the regions we’re smoothing out.

This is because of the agent’s quasilinear utilities: changing $x$, and $t$, for some $\theta$ affects required payoffs for all $\theta$’s equally, so does not affect local incentives.