Reputation

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Game with Short-Run Players

\((N, A, u)\): two-player normal-form game played in every period \(t = 0, 1, \ldots\)

1 is a long-run player and 2 is a short-run player (series of one-period players or a very impatient player). 2 plays a best response to 1’s anticipated action at every date.

Fudenberg, Kreps, and Maskin (1988): folk theorem if game is common knowledge

- \(B_2\): 2’s mixed best responses in stage game to 1’s mixed actions
- \(\underline{u}_1 = \min_{\sigma_2 \in B_2} \max_{a_1 \in A_1} u_1(a_1, \sigma_2)\)
- Any payoff for player 1 above \(\underline{u}_1\) is sustainable in a subgame perfect equilibrium for high \(\delta\)

Fudenberg and Levine (1989): if game is perturbed to allow for irrational types of player 1, folk theorem overturned

- \(u_1^* = \max_{a_1 \in A_1} \min_{\sigma_2 \in BR_2(a_1)} u_1(a_1, \sigma_2)\): Stackelberg payoff
  - 1 obtains his Stackelberg payoff in any Nash equilibrium for high \(\delta\)

Compare \(\underline{u}_1\) and \(u_1^*\) for Cournot duopoly.
Perturbed Game

- $\Omega$: countable space of types for player 1, prior $\mu$
- Only player 1 knows his type
- $u_1(a, \omega)$: player 1’s payoff depends on $\omega$; player 2’s does not
- $\omega_0$: “rational” type of player 1 with payoffs given by original $u_1$
- $\omega(a_1)$: “crazy” type of player 1 for which playing $a_1$ at every history is a strictly dominant strategy in the repeated game
- $\omega^* = \omega(a_1^*)$ with $\mu(\omega^*) > 0$
Key Lemma

- Any strategy profile $\sigma$ (together with $\mu$) generates a unique joint distribution over play paths and types $\pi \in \Delta((A_1 \times A_2)^\infty \times \Omega)$
- $h^*$: event in $(A_1 \times A_2)^\infty \times \Omega$ in which $a_1^t = a_1^*$ for all $t$
- $\pi_t^* = \pi(a_1^t = a_1^*|h^{t-1})$: probability of $a_1^*$ at $t$ conditional on history $h^{t-1}$
- $n(\pi_t^* \leq \bar{\pi})$: number of periods $t$ s.t. $\pi_t^* \leq \bar{\pi}$ for $\bar{\pi} \in (0, 1)$
- $\pi_t^*$ and $n(\pi_t^* \leq \bar{\pi})$ are random variables defined on path-type space

Lemma 1

Let $\sigma$ be a strategy profile such that $\pi(h^*|\omega^*) = 1$. Then

$$\pi \left( n(\pi_t^* \leq \bar{\pi}) \leq \frac{\ln \mu^*}{\ln \bar{\pi}} \right| h^* \right) = 1.$$
Proof

$h^t$: history of length $t$ with $\pi(h^t) > 0$ in which player 1 played $a_1^*$ every period

$h^{t,1}$ ($h^{t,2}$): event that $h^{t-1}$ is observed and player 1 (2) plays at $t$ as in $h^t$

$$
\pi(\omega^*|h^t) = \frac{\pi(h^t \& \omega^*|h^{t-1})}{\pi(h^t|h^{t-1})} = \frac{\pi(\omega^*|h^{t-1})\pi(h^t|\omega^*, h^{t-1})}{\pi(h^t|h^{t-1})} = \frac{\pi(\omega^*|h^{t-1})\pi(h^{t,1}|\omega^*, h^{t-1})\pi(h^{t,2}|\omega^*, h^{t-1})}{\pi(h^{t,1}|h^{t-1})\pi(h^{t,2}|h^{t-1})} = \frac{\pi(\omega^*|h^{t-1})\pi(h^{t,2}|\omega^*, h^{t-1})}{\pi(h^{t,1}|h^{t-1})\pi(h^{t,2}|h^{t-1})} = \frac{\pi(\omega^*|h^{t-1})}{\pi_t^*}
$$
\[ \pi(\omega^*|h^t) = \frac{\pi(\omega^*|h^{t-1})}{\pi_t^*} = \ldots = \frac{\pi(\omega^*|h^0)}{\pi_t^*\pi_{t-1}^* \cdots \pi_0^*} = \frac{\mu^*}{\pi_t^*\pi_{t-1}^* \cdots \pi_0^*} \]

Since \( \pi(\omega^*|h^t) \leq 1 \), at most \( \ln \mu^*/\ln \bar{\pi} \) terms in the denominator of the last expression can be \( \leq \bar{\pi} \).

Therefore, with probability 1,

\[ n(\pi_t^* \leq \bar{\pi}) \leq \ln \mu^*/\ln \bar{\pi}. \]
Main Result

- \( u_m = \min_{\sigma_2} u_1(a_1^*, \sigma_2, \omega_0) \): lowest stage payoff for 1 when he plays \( a_1^* \)
- \( u_M = \max_a u_1(a, \omega_0) \): highest stage payoff for 1
- \( \bar{u}_1 = \max_{a_1} \max_{\sigma_2 \in BR_2(a_1)} u_1(a_1, a_2) \): “upper” Stackelberg payoff
- \( v_1(\delta, \mu, \omega_0) (\bar{v}_1(\delta, \mu, \omega_0)) \): infimum (supremum) of 1’s payoffs in repeated game across Nash equilibria in which 1 uses a pure strategy

Theorem 1

For any value \( \mu^* \), there exists a number \( \kappa(\mu^*) \) s.t. for all \( \delta \) and all \( (\mu, \Omega) \) with \( \mu(\omega^*) = \mu^* \), we have

\[
\bar{v}_1(\delta, \mu, \omega_0) \geq \delta^{\kappa(\mu^*)} u_1^* + (1 - \delta^{\kappa(\mu^*)}) u_m.
\]

Moreover, there exists \( \kappa \) such that for all \( \delta \), we have

\[
\bar{v}_1(\delta, \mu, \omega_0) \leq \delta^{\kappa} \bar{u}_1 + (1 - \delta^{\kappa}) u_M.
\]

As \( \delta \to 1 \), the payoff bounds converge to \( u_1^* \) and \( \bar{u}_1 \) (generically identical).
Proof

\( \exists \pi < 1 \) s.t. in any Nash equilibrium player 2 plays a best response to \( a_1^* \) at every stage \( t \) where \( \pi^*_t > \pi \)

- Pure strategy best response correspondence has closed graph.
- Action spaces are finite.

\( \exists \kappa(\mu^*) \) s.t. \( \pi(n(\pi^* \leq \pi) > \kappa(\mu^*) \mid h^*) = 0 \) (by the lemma)

If rational player 1 deviates to playing \( a_1^* \) always, there are at most \( \kappa(\mu^*) \) periods in which player 2 will not play a best response to \( a_1^* \). Then payoff from deviating is at least

\[
\delta^{\kappa(\mu^*)} u_1^* + (1 - \delta^{\kappa(\mu^*)}) u_m.
\]

Proof for upper bound requires a version of the lemma for \( \omega_0 \ldots \) from the perspective of rational player 1, player 2 plays a best response to his action at all but a finite set of dates.

Fudenberg and Levine (1992): extension to mixed strategy Nash equilibria