Example 1. Consider the following simple function

\[ h(x) = -x^3 + x \]

then clearly \( \bar{x} = 0 \) is the unique steady state of \( x_{t+1} = h(x_t) \). It is also globally stable. This follows since \( -x^3 < 0 \) for \( x > 0 \) and \( -x^3 > 0 \) for \( x < 0 \) so that \( h(x) = -x^3 + x < x \) for \( x > 0 \) and \( h(x) = -x^3 + x > x \) for \( x < 0 \). Thus \( x \) is rising if \( x \) is below 0 and falling if it is above 0. The convergence is then monotonic.

However note that at \( \bar{x} = 0 \) we have that \( h'(0) = 1 \) so that \( A = 1 \) and the eigenvalue is \( \lambda = 1 \), thus \( |\lambda| = 1 \).

One may oppose this example since we were requiring \( I - A \) to be non-singular, here \( I - A = 0 \) so it is singular. The next example shows a case with \( I - A \) non-singular.

Example 2. Take

\[ h(x) = x^3 - x \]

it is easy to see that \( \bar{x} = 0 \) is the unique steady state of \( x_{t+1} = h(x_t) \) for \( x \in [-1, 1] \). It is easy to see that the system is locally stable around \( \bar{x} \) (it is not monotonic though).

However note that at \( \bar{x} = 0 \) we have that \( h'(0) = -1 \) so that \( A = -1 \) and the eigenvalue is \( \lambda = -1 \), thus \( |\lambda| = 1 \). Note that in this case \( I - A = -2 \) is singular.

Note: Clearly an eigenvalue with absolute value of 1 does not ensure local convergence, just take \( h(x) = x^3 + x \) or \( h(x) = -x^3 - x \) for example.

Remarks: Of course both of these policy functions can be generated as optimal policy functions for some concave \( F \) and some \( 0 < \beta < 1 \) using the Boldrin-Montrucchio construction argument we went over in class. Thus these point are of interest for us, they can arise in applications.

We conclude from these 2 examples that a one dimensional system may be stable even if we don’t have \( |\lambda| < 1 \), if we do have \( |\lambda| = 1 \). More generally, with more dimensions this point may affect the dimensionality of the subset of the neighbourhood over which the system is stable. That is, even if we have \( |\lambda_i| < 1 \) for only \( m \) eigenvalues, if we have some other eigenvalues with \( |\lambda_i| = 1 \) we may [we can’t be sure, see the ”note” above] have convergence starting from \( x_0 \) belonging to a subset of greater dimension than \( m \).