1 Solutions Pset 3

1) Do some programing
   3) Brock Mirman problem
      a) Take \( V = a_1 \log k + a_2 \log \theta + a_3 \). Then the max problem is

\[
TV (k) = \max_{0 \leq k' \leq Ak^\alpha \theta} \ln (Ak^\alpha \theta - k') + \beta E_\theta [a_1 \log k' + a_2 \log \theta + a_3]
\]

\[
TV (k) = \max_{0 \leq k' \leq Ak^\alpha \theta} \ln (Ak^\alpha \theta - k') + \beta a_1 \log k' + \beta a_2 E_\theta \log \theta + \beta a_3
\]

The FOC condition for this problem is (assuming interior),

\[
- \frac{1}{Ak^\alpha \theta - k'} + \frac{\beta a_1}{k'} = 0
\]

Which implies that

\[
k' = \frac{\beta a_1}{1 + \beta a_1} Ak^\alpha \theta
\]

And

\[
TV (k) = \ln \left( \frac{1}{1 + \beta a_1} Ak^\alpha \theta \right) + \beta a_1 \log \frac{\beta a_1}{1 + \beta a_1} Ak^\alpha \theta + \beta a_2 E_\theta \log \theta + \beta a_3
\]

\[
= \alpha (1 + \beta a_1) \log k + (1 + \beta a_1) \log \theta + \beta a_2 E_\theta \log \theta + \beta a_3
\]

Given that \( V (k) = a_1 \log k + a_2 \log \theta + a_3 \), using \( V (k) = TV (k) \) we have that

\[
a_1 = \alpha (1 + \beta a_1)
\]

\[
a_2 = 1 + \beta a_1
\]

\[
a_3 = \beta a_1 \log \frac{A \beta a_1}{1 + \beta a_1} + \ln \left( \frac{A}{1 + \beta a_1} \right) + \beta a_2 E_\theta \log \theta + \beta a_3
\]

So,

\[
a_1 = \frac{\alpha}{1 - \beta \alpha} > 0
\]

\[
a_2 = \frac{1}{1 - \beta \alpha}
\]
And $a_3$ is given by
\[ a_3 = \frac{1}{1 - \beta} \left[ \beta a_1 \log \frac{A \beta a_1}{1 + \beta a_1} + \ln \left( \frac{A}{1 + \beta a_1} \right) + \beta a_2 E \log \theta \right] \]
The proof that $V = V^*$ is done in page 275,276 of SLP.

b) The optimal rule for consumption is then
\[ c(k, \theta) = A^k \theta - k'(k, \theta) = \]
\[ c(k, \theta) = \frac{1}{1 + \beta a_1} A^k \theta \]
\[ c(k, \theta) = (1 - \beta \alpha) A^k \theta \]

So, we have that
\[ \frac{\partial c}{\partial \beta} < 0 \]
and
\[ \frac{\partial c}{\partial \alpha} = \alpha (1 - \beta \alpha) A^{k - 1} \theta - \beta A^k \theta \]
\[ = [\alpha (1 - \beta \alpha) - \beta k] A^{k - 1} \theta \]

There are two effects, depending on the level of $k$.

c) You can do it ex-ante (before the value of $\theta$ is realized), then
\[ V(k) = \int \left( \max_{0 < k' \leq A^k \theta} \left\{ \ln (A^k \theta - k') + \beta V[k'] \right\} \right) h(\theta) \, d\theta \]

4) a) The main conflict is the change in preferences. They value consumption paths differently because they discount the future in different ways. In particular, time-$t$ self values consumption at time-$t$ versus time-$(t + 1)$ more than any time-$\tau$ self with $\tau < t$, as long as $\beta < 1$. For $\beta = 1$ they all agree.

b) Every self maximizes its utility subject to what other types will do in the future. So,
\[ V(k_0) = \max_c u(c) + \delta W(k_1) \quad (1) \]

Where $\delta W(k)$ is the discounted value for todays self of leaving $k'$ for the future. So,
\[ W(k_t) = \beta \sum_i \delta^i u(c^*(k_{t+i})) \]
Where \( c^* (k_{t+1}) \) is the optimal consumption rule that future selves will follow (we are assuming symmetry, and hence \( c^* \) is time-independent). Now take (??) and do the following:

\[
V (k_0) = u (c_0^*) + \delta W (k_1) = u (c_0^*) + \beta \delta \sum_i \delta^i u (c^* (k_{t+i}))
\]

\[
V (k_0) - (1 - \beta) u (c_0^*) = \beta u (c_0^*) + \beta \delta \sum_i \delta^i u (c^* (k_{t+i}))
\]

\[
V (k_0) - (1 - \beta) u (c_0^*) = W (k_0)
\]

So, We can define \( W \) recursevly as

\[
W (k) = V (k) - (1 - \beta) u (c^* (k))
\]

\[
W (k) = \max_c \{ u (c) + \delta W (f (k) - c) \} - (1 - \beta) u (c^* (k))
\]

The \( T \) operator is such that \( TW (k) = \max_c \{ u (c) + \delta W (f (k) - c) \} - (1 - \beta) u (c^* (k)) \) and we are looking for a fixed point of \( T \).

c) If \( \beta = 1 \), you can easily show that \( T \) is a contraction mapping (is monotone and satisfies discounting). This means that there is a unique \( W \) that solves the functional equation, and unique Markov equilibrium.

d) If \( \beta < 1 \) the \( T \) operator satisfies discounting:

\[
T (W (k) + a) = \max_c \{ u (c) + \delta (W (f (k) - c) + a) \} - (1 - \beta) u (c^* (k))
\]

\[
= \max_c \{ u (c) + \delta W (f (k) - c) \} - (1 - \beta) u (c^* (k)) + \delta a
\]

\[
= TW (k) + \delta a
\]

It does not however, necessarily satisfies monotonicity. Higher \( W \), might imply higher \( c^* (k) \) for some capital level, and hence \( \max_c \{ u (c) + \delta (W (f (k) - c) + a) \} - (1 - \beta) u (c^* (k)) \) might not increase.

e) If \( u = \log c \) and \( f = Ak^\alpha \), then we can do part 3.

\[
TW (k) = \max_c \{ u (c) + \delta W (f (k) - c) \} - (1 - \beta) u (c^* (k))
\]

\[
= \max_c \{ \log c + \delta a \log (Ak^\alpha - c) + \delta b \} - (1 - \beta) u (c^* (k))
\]

\[
c^* (k) : \quad \frac{1}{c} = \frac{\delta a}{Ak^\alpha - c}
\]

\[
c = \frac{1}{1 + \delta a} Ak^\alpha
\]
So,

\[ TW(k) = \log \frac{1}{1+\delta a} Ak^\alpha + \delta a \log \left( Ak^\alpha - \frac{1}{1+\delta a} Ak^\alpha \right) + \delta b - (1-\beta) \log \frac{1}{1+\delta a} Ak^\alpha \]

\[ = \log \frac{1}{1+\delta a} A + \alpha \log k + \delta A \log \frac{\delta a}{1+\delta a} A + \alpha \delta a \log k + \delta b - (1-\beta) \log \frac{1}{1+\delta a} A - (1-\beta) \alpha \log k \]

\[ = \alpha [(1+\delta a) - (1-\beta)] \log k + \delta b + \log \frac{1}{1+\delta a} A + \delta A \log \frac{\delta a}{1+\delta a} A - (1-\beta) \log \frac{1}{1+\delta a} A \]

So,

\[ a = \frac{\alpha \beta}{1-\alpha \delta} \]

And you can easily compute \( b \).

The equilibrium consumption policy is then

\[ c = \frac{1-\alpha \delta}{1-\alpha \delta (1-\beta)} Ak^\alpha \]

Higher \( \beta \) implies higher consumption (the impatience has decreased).  

f) For \( \beta = 0 \) we have that

\[ \tilde{c} = (1-\alpha \delta^e) Ak^\alpha \]

So we need \( \tilde{\delta} \) to be such that

\[ \frac{1}{1-\alpha \delta (1-\beta)} \beta \delta = \delta^e \]

Now

\[ \delta > \delta^e > \beta \delta \]

given that \( \beta < 1 \).

A hyperbolic consumer looks like an exponential with an appropriate discount rate!!.
a contradiction with $\beta > 0$.

**c.** The proof parallels the argument in **b.**

**Exercise 6.7**

**a.** Actually, Assumption 4.9 is not needed for uniqueness of the optimal capital sequence.

A4.3: $K = [0, 1] \subseteq \mathbb{R}^l$ and the correspondence

$$\Gamma(k) = \{ y : y \in K \}$$

is clearly compact-valued and continuous.

A4.4: $F(k, y) = (1 - y)^{(1 - \theta)\alpha}k^{\theta\alpha}$ is clearly bounded in $K$, and it is also continuous. Also, $0 \leq \beta \leq 1$.

A4.7: Clearly $F$ is continuously differentiable, then

$$F_k = \theta\alpha(1 - y)^{(1 - \theta)\alpha}k^{\theta\alpha - 1}$$
$$F_y = -(1 - \theta)\alpha(1 - y)^{(1 - \theta)\alpha - 1}k^{\theta\alpha}$$
$$F_{kk} = \theta\alpha(1 - y)(\theta\alpha - 1)(1 - \theta)^{\alpha - 1}k^{\theta\alpha - 2} < 0$$
$$F_{yy} = (1 - \theta)\alpha[(1 - \theta)\alpha - 1](1 - y)^{(1 - \theta)\alpha - 2}k^{\theta\alpha} < 0$$
$$F_{xy} = -\theta\alpha(1 - \theta)\alpha(1 - y)^{(1 - \theta)\alpha - 1}k^{\theta\alpha - 1} < 0,$$

and $F_{kk}F_{yy} - F_{xy}^2 > 0$, hence $F$ is strictly concave.

A4.8: Take two arbitrary pairs $(k, y)$ and $(k', y')$ and $0 < \pi < 1$. Define $k^\pi = \pi k + (1 - \pi)k'$, $y^\pi = \pi y + (1 - \pi)y'$. Then, since $\Gamma(k) = \{ y : 0 \leq y \leq 1 \}$ for all $k$ it follows trivially that if $y \in \Gamma(k)$ and $y' \in \Gamma(k')$ then $y^\pi \in \Gamma(k^\pi) = \Gamma(k) = \Gamma(k') = K$.

A4.9: Define $A = K \times K$ as the graph of $\Gamma$. Hence $F$ is continuously differentiable because $U$ and $f$ are continuously differentiable. The Euler equation is

$$\alpha(1 - \theta)(1 - k_{t+1})^{(1 - \theta)\alpha - 1}k_t^{\theta\alpha} = \beta\alpha\theta(1 - k_{t+2})^{(1 - \theta)\alpha}k_{t+1}^{\theta\alpha - 1}.$$
b. Evaluating the Euler equation at $k_{t+1} = k_t = k^*$, we get

$$(1 - \theta)k^* = \beta \theta (1 - k^*),$$

or

$$k^* = \frac{\beta \theta}{1 - \theta + \beta \theta}.$$ 

c. From the Euler equation, define

$$W(k_t, k_{t+1}, k_{t+2}) \equiv \alpha (1 - \theta) (1 - k_{t+1})^{(1-\theta)\alpha-1} k_t^{\theta \alpha}$$

$$- \beta \alpha \theta (1 - k_{t+2})^{(1-\theta)\alpha} k_{t+1}^{\theta \alpha-1} = 0.$$

Hence, expanding $W$ around the steady state

$$W(k_t, k_{t+1}, k_{t+2}) = W(k^*, k^*, k^*) + W_1(k^*) (k_t - k^*)$$

$$+ W_2(k^*) (k_{t+1} - k^*) + W_3(k^*) (k_{t+2} - k^*),$$

where

$$W_1(k^*) = \alpha^2 (1 - \theta) \theta (1 - k^*)^{(1-\theta)\alpha-1} (k^*)^{\theta \alpha-1},$$

$$W_2(k^*) = -\alpha (1 - \theta) [(1 - \theta) \alpha - 1] (1 - k^*)^{(1-\theta)\alpha-2} (k^*)^{\theta \alpha}$$

$$- \beta \alpha \theta (\theta \alpha - 1) (1 - k^*)(1-\theta)\alpha (k^*)^{\theta \alpha-2},$$

$$W_3(k^*) = \beta \alpha^2 (1 - \theta) (1 - k^*)^{(1-\theta)\alpha-1} (k^*)^{\theta \alpha-1}.$$

Normalizing by $W_3(k^*)$ and using the expression obtained for the steady state capital we finally get

$$\beta^{-1} (k_t - k^*) + B (k_{t+1} - k^*) + (k_{t+2} - k^*) = 0,$$

where

$$B = \frac{1 - \alpha (1 - \theta)}{\alpha (1 - \theta)} + \frac{1 - \alpha \theta}{\alpha \theta \beta}.$$ 

That both of the characteristic roots are real comes from the fact that the return function satisfies Assumptions 4.3-4.4 and 4.7-4.9 and it is twice differentiable, so the results obtained in Exercise 6.6 apply.
To see that $\lambda_1 = (\beta \lambda_2)^{-1}$ it is straightforward from the fact that

$$\lambda_1 \lambda_2 = \left( \frac{(-B) + \sqrt{B^2 - 4\beta^{-1}}}{2} \right) \left( \frac{(-B) - \sqrt{B^2 - 4\beta^{-1}}}{2} \right)$$

$$= \frac{(-B)^2 - (B^2 - 4\beta^{-1})}{4} = \beta^{-1}.$$ 

To see that $\lambda_1 + \lambda_2 = -B$, just notice that

$$\lambda_1 + \lambda_2 = \frac{(-B) + \sqrt{B^2 - 4\beta^{-1}}}{2} + \frac{(-B) - \sqrt{B^2 - 4\beta^{-1}}}{2} = -B.$$ 

Then, $\lambda_1 \lambda_2 > 0$ and $\lambda_1 + \lambda_2 < 0$ implies that both roots are negative.

In order to have a locally stable steady state $k^*$ we need one of the characteristic roots to be less than one in absolute value. Given that both roots are negative, this implies that we need $\lambda_1 > -1$, or

$$-B + \sqrt{B^2 - 4\beta^{-1}} > -2,$$

which after some straightforward manipulation implies

$$B > \frac{1 + \beta}{\beta}.$$ 

Substituting for $B$ we get

$$\frac{1 - \theta + \theta \beta}{2\theta(1 + \beta)(1 - \theta)} > \alpha,$$

or equivalently

$$\beta > \frac{(2\theta \alpha - 1)(1 - \theta)}{|1 - 2\alpha(1 - \theta)| \theta}.$$ 

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d. To find that $k^* = 0.23$, evaluate the equation for $k^*$ obtained in b. at the given parameter values. To see that $k^*$ is unstable, evaluate $\lambda_1$ at the given parameter values. Notice also that those parameter values do not satisfy the conditions derived in c.
e. Note that since $F$ is bounded, the two-cycle sequence satisfies the transversality conditions
\[
\lim_{t \to \infty} \beta^t F_1(x, y) \cdot x = 0 \quad \text{and} \quad \lim_{t \to \infty} \beta^t F_1(x, y) \cdot y = 0,
\]
for any two numbers $x, y \in [0, 1]$, $x \neq y$. Hence, by Theorem 4.15, if the two cycle $(x, y)$ satisfies
\[
F_y(x, y) + \beta F_x(y, x) = 0 \quad \text{and} \quad F_y(y, x) + \beta F_x(x, y) = 0,
\]
it is an optimal path.

Conversely, if $(x, y)$ is optimal and the solution is interior, then it satisfies
\[
F_y(x, y) + \beta \nu'(y) = 0 \quad \text{and} \quad \nu'(y) = F_x(y, x),
\]
\[
F_y(y, x) + \beta \nu'(x) = 0 \quad \text{and} \quad \nu'(x) = F_x(x, y),
\]
and hence it satisfies the Euler equations stated in the text.

Notice that the pair $(x, y)$ defining the two-cycle should be restricted to the open interval $(0, 1)$.

f. We have that
\[
F_y(x, y) + \beta F_x(y, x) = \beta \alpha \theta y^{\alpha \theta - 1}(1 - x)^{\alpha (1 - \theta)} - \alpha (1 - \theta) x^{\alpha \theta} (1 - y)^{\alpha (1 - \theta) - 1},
\]
and
\[
F_y(y, x) + \beta F_x(x, y) = \beta \alpha \theta x^{\alpha \theta - 1}(1 - y)^{\alpha (1 - \theta)} - \alpha (1 - \theta) y^{\alpha \theta} (1 - x)^{\alpha (1 - \theta) - 1}
\]

The pair $(0.25, 0.18)$ the above set of equations equal to zero, and from the result proved in part e. we already know this is a necessary and sufficient condition for the pair to be a two-cycle.
Define
\[ E^1(k_t, k_{t+1}, k_{t+2}, k_{t+3}) = -\alpha(1 - \theta)k_{t+1}\alpha^\theta(1 - k_{t+2})^{\alpha(1-\theta) - 1} \]
\[ + \beta\alpha\theta k_{t+2}^{\alpha^\theta - 1}(1 - k_{t+3})^{\alpha(1-\theta)} \]
\[ = -\alpha(1 - \theta)x^{\alpha^\theta}(1 - y)^{\alpha(1-\theta) - 1} \]
\[ + \beta\alpha\theta y^{\alpha^\theta - 1}(1 - x)^{\alpha(1-\theta)} \]
\[ = 0 \]
\[ E^2(k_t, k_{t+1}, k_{t+2}, k_{t+3}) = -\alpha(1 - \theta)k_{t+1}\alpha^\theta(1 - k_{t+3})^{\alpha(1-\theta) - 1} \]
\[ + \beta\alpha\theta k_{t+2}^{\alpha^\theta - 1}(1 - k_{t+3})^{\alpha(1-\theta)} \]
\[ = -\alpha(1 - \theta)y^{\alpha^\theta}(1 - x)^{\alpha(1-\theta) - 1} \]
\[ + \beta\alpha\theta x^{\alpha^\theta - 1}(1 - y)^{\alpha(1-\theta)} \]
\[ = 0. \]

Let \( E^i \) be the derivative of \( E^j \) with respect to the \( i \)th argument.

Then, the derivatives are
\[ E_1^1 = 0, \]
\[ E_2^1 = -\alpha^2\theta(1 - \theta)x^{\alpha^\theta - 1}(1 - y)^{\alpha(1-\theta) - 1}, \]
\[ E_3^1 = -\alpha(1 - \theta)x^{\alpha^\theta}[\alpha(1 - \theta) - 1](1 - y)^{\alpha(1-\theta) - 2} \]
\[ + \beta\alpha\theta(\alpha - 1)y^{\alpha^\theta - 2}(1 - x)^{\alpha(1-\theta)}, \]
\[ E_4^1 = \beta\alpha\theta y^{\alpha^\theta - 1}(1 - x)^{\alpha(1-\theta) - 1}, \]
\[ E_1^2 = -\alpha^2\theta(1 - \theta)y^{\alpha^\theta - 1}(1 - x)^{\alpha(1-\theta) - 1}, \]
\[ E_2^2 = -\alpha(1 - \theta)y^{\alpha^\theta}[\alpha(1 - \theta) - 1](1 - x)^{\alpha(1-\theta) - 2} \]
\[ + \beta\alpha\theta(\alpha - 1)x^{\alpha^\theta - 2}(1 - y)^{\alpha(1-\theta)}, \]
\[ E_3^2 = \beta\alpha\theta x^{\alpha^\theta - 1}(1 - y)^{\alpha(1-\theta) - 1}, \]
\[ E_4^2 = 0. \]

Using the fact that \( k_{t+2} = k_t \) in \( E_1 \), expand this system around \((0.29, 0.18)\). Denoting by \( \hat{K} \) deviations around the stationary point \( \hat{K} \), we can express the linearized system as
\[ \hat{K}_{t/2+1} = \left[ \begin{array}{c} \hat{k}_{t+3} \\ \hat{k}_{t+2} \end{array} \right] = \hat{H} \left[ \begin{array}{c} \hat{k}_{t+1} \\ \hat{k}_t \end{array} \right] = \hat{H}\hat{K}_{t/2} \]

where
\[ \hat{H} = \begin{bmatrix} E_1^1 & 0 \\ 0 & E_3^2 \end{bmatrix}^{-1} \begin{bmatrix} E_1^2 & E_1^1 \\ E_2^2 & E_2^1 \end{bmatrix} \]

evaluate this to get stability.