Recursive Methods
Outline Today’s Lecture

- continue APS: worst and best value
- Application: Insurance with Limited Commitment
- stochastic dynamics
**B(W) operator**

**Definition:** For each set $W \subset R$, let $B(W)$ be the set of possible values $\omega = (1 - \delta)r(x, y) + \delta \omega_1$ associated with some admissible tuples $(x, y, \omega_1, \omega_2)$ wrt $W$:

$$B(W) \equiv \left\{ w : \exists (x, y) \in C \text{ and } \omega_1, \omega_2 \in W \text{ s.t.} \right.$$

$$(1 - \delta)r(x, y) + \delta \omega_1 \geq (1 - \delta)r(x, \hat{y}) + \delta \omega_2, \forall \hat{y} \in Y \left. \right\}$$

- note that $V$ is a fixed point $B (V) = V$
- actually, $V$ is the biggest fixed point

[fixed point not necessarily unique!]

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Finding $V$

In this simple case here we can do more...

- lowest $v$ is self-enforcing
  - highest $v$ is self-rewarding

$$v_{low} = \min_{(x,y)\in C} \{ (1 - \delta) r(x, y) + \delta v \}$$

$$(1 - \delta) r(x, y) + \delta v \geq (1 - \delta) r(x, \hat{y}) + \delta v_{low} \text{ all } \hat{y} \in Y$$

then

$$\Rightarrow v_{low} = (1 - \delta) r(h(y), y) + \delta v \geq (1 - \delta) r(h(y), H(h(y))) + \delta v_{low}$$

- if binds and $v > v_{low}$ then minimize RHS of inequality

$$v_{low} = \min_y r(h(y), H(h(y)))$$
**Best Value**

- for Best, use Worst to punish and Best as reward

solve:

\[
\max_{(x, y) \in C, v \in V} \left\{ (1 - \delta) r(x, y) + \delta v_{\text{high}} \right\}
\]

\[
(1 - \delta)r(x, y) + \delta v_{\text{high}} \geq (1 - \delta)r(x, \hat{y}) + \delta v_{\text{low}} \quad \text{all } \hat{y} \in Y
\]

then clearly \( v_{\text{high}} = r(x, y) \)

- so

\[
\max r(h(y), y)
\]

subject to \( r(h(y), y) \geq (1 - \delta)r(h(y), H(h(y))) + \delta v_{\text{low}} \)

- if constraint not binding \( \rightarrow \) Ramsey (first best)

- otherwise value is constrained by \( v_{\text{low}} \)
Insurance with Limited Commitment

• 2 agents utility $u(c^A)$ and $u(c^B)$

• $y_t^A$ is iid over $[y_{low}, y_{high}]$

• $y_t^B = \bar{y} - y_t^A$ same distribution as $y_t^A$ (symmetry)

• define

$$w_{aut} = \frac{E u(y)}{1 - \beta}$$

• let $[w_l(y), w_h(y)]$ be the set of attainable levels of utility for $A$ when $A$ has income $y$ (by symmetry it is also that of $A$ with income $\bar{y} - y$)

• $v(w, y)$ for $w \in [w_l, w_h]$ be the highest utility for $B$ given that $A$ is promised $w$ and has income $y$ (the pareto frontier)
Recursive Representation

\[ v(w, y) = \max\{u(c^B) + \beta Ev(w'(y'), y')\} \]

\[ w = u(c^A) + \beta Ew(y') \]

\[ u(c^A) + \beta Ew(y') \geq u(y) + \beta v_{aut} \]

\[ u(c^B) + \beta Ev(w'(y'), y') \geq u(\bar{y} - y) + \beta v_{aut} \]

\[ c^A + c^B \leq \bar{y} \]

\[ w'(y') \in [w_l(y'), w_h(y')] \]

- is this a contraction? NO
- is it monotonic? YES
- should solve for \([w_l(y), w_h(y)]\) jointly
  - clearly \(w_l(y) = u(y) + \beta v_{aut}\)
  - \(w_h(y)\) such that \(v(w_h(y), y) = u(\bar{y} - y) + \beta v_{aut}\)
Stochastic Dynamics

- output of stochastic dynamic programming:
  
  optimal policy:

  \[ x_{t+1} = g(x_t, z_t) \]

- convergence to steady state?
  on rare occasions (but not necessarily never...)

- convergence to something?
Notion of Convergence

Idea:

- start at $t = 0$ with some $x_0$ and $s_0$
- compute $x_1 = g(x_0, z_0) \rightarrow x_1$ is not uncertain from $t = 0$ view
- $z_1$ is realized $\rightarrow$ compute $x_2 = g(x_1, z_1)$
  $x_2$ is random from point of view of $t = 0$
- continue... $x_3, x_4, x_5,...x_t$ are random variables from $t = 0$ perspective
- $F_t(x_t)$ distribution of $x_t$ (given $x_0, z_0$)
  more generally think of joint distribution of $(x, z)$
- convergence concept
  $$\lim_{t \to \infty} F_t(x) = F(x)$$
Examples

- stochastic growth model
- Brock-Mirman ($\delta = 0$)

\[ u(c) = \log c \]
\[ f(A, k) = Ak^\alpha \]

and $A_t$ is i.i.d. optimal policy

\[ k_{t+1} = sA_t k_t^\alpha \]

with $s = \beta \alpha$
Examples

- search model: last recitation
  employment state $u$ and $e$ (also wage if we want)
  $\rightarrow$ invariant distribution gives steady state unemployment rate

- if uncertainty is idiosyncratic in a large population
  $\Rightarrow F$ can be interpreted as a cross section
income fluctuations problem

\[ v(a, y; R) = \max_{0 \leq a' \leq Ra + y} \{ u(Ra + y - a') + \beta E [v(a', y'; R) | y] \} \]

solution \( a' = g(a, y; R) \)

invariant distribution \( F(a; R) \)

cross section assets in large population

how does \( F \) vary with \( R \)? (continuously?)

once we have \( F \) can compute moments:

market clearing

\[ \int a \, dF(a; R) = K \]
Markov Chains

• $N$ states of the world
• let $\Pi_{ij}$ be probability of $s_{t+1} = j$ conditional on $s_t = i$
• $\Pi = (\Pi_{ij})$ transition matrix
• $p$ distribution over states
• $p_0 \rightarrow p_1 = \Pi p_0$ (why?) $\rightarrow \ldots \rightarrow$

$$p_t = \Pi^t p_0$$

• does $\Pi^t$ converge?
Examples

- example 1: $\Pi^t$ converges
- example 2: transient state
- example 3: $\Pi^t$ does not converge but fluctuates
- example C: ergodic sets
Theorem

Let \( S = \{s_1, \ldots, s_l\} \) and \( \Pi \)

a. \( S \) can be partitioned into \( M \) ergodic sets
b. the sequence

\[
\left( \frac{1}{n} \right) \sum_{k=0}^{n-1} \Pi^k \to Q
\]

c. each row of \( Q \) is an invariant distribution and so are the convex combinations
Theorem
Let $S = \{s_1, \ldots, s_l\}$ and $\Pi$ then $\Pi$ has a unique ergodic set if and only if there is a state $s_j$ such that for all $i$ there exists an $n \geq 1$ such that $\pi_{ij}^{(n)} > 0$. In this case $\Pi$ has a unique invariant distribution $p^*$; each row of $Q$ equals $p^*$.

Theorem
Let $\varepsilon_{ij}^n = \min_i \pi_{ij}^n$ and $\varepsilon^n = \sum_j \varepsilon_{ij}^n$. Then $S$ has a unique ergodic set with no cyclical moving subsets if and only if for some $N \geq 1 \varepsilon^N > 0$. In this case $\Pi^n \to Q$. 

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