Recursive Methods
Outline Today’s Lecture

• housekeeping: ps#1 and recitation day/ theory general / web page

• finish Principle of Optimality:
  Sequence Problem $\iff$ solution to Bellman Equation
  (for values and plans)

• begin study of Bellman equation with bounded and continuous $F$

• tools: contraction mapping and theorem of the maximum
**Sequence Problem vs. Functional Equation**

- **Sequence Problem: (SP)**

\[
V^* (x_0) = \sup \sum_{\{x_{t+1}\}_{t=0}^{\infty}} \beta^t F(x_t, x_{t+1}) \\
\text{s.t. } x_{t+1} \in \Gamma(x_t) \\
x_0 \text{ given}
\]

- ... more succinctly

\[
V^* (x_0) = \sup_{\tilde{x} \in \Pi(x_0)} u(\tilde{x}) \quad (SP)
\]

- **Functional equation (FE) [this particular FE called Bellman Equation]**

\[
V(x) = \max_{y \in \Gamma(x)} \{F(x, y) + \beta V(y)\} \quad (FE)
\]
Principle of Optimality

IDEA: use BE to find value function $V^*$ and optimal plan $x^*$

- **Thm 4.2.** $V^*$ defined by SP $\Rightarrow V^*$ solves FE
- **Thm 4.3.** $V$ solves FE and ...... $\Rightarrow V = V^*$
- **Thm 4.4.** $\tilde{x}^* \in \Pi (x_0)$ is optimal
  $\Rightarrow V^* (x^*_t) = F (x^*_t, x^*_t+1) + \beta V^* (x^*_t+1)$
- **Thm 4.5.** $\tilde{x}^* \in \Pi (x_0)$ satisfies $V^* (x^*_t) = F (x^*_t, x^*_t+1) + \beta V^* (x^*_t+1)$ and ......
  $\Rightarrow \tilde{x}^*$ is optimal
Why is this Progress?

- **intuition**: breaks planning horizon into two: ‘now’ and ‘then’
- **notation**: reduces unnecessary notation (especially with uncertainty)
- **analysis**: prove existence, uniqueness and properties of optimal policy (e.g. continuity, monotonicity, etc...)
- **computation**: associated numerical algorithm are powerful for many applications
Proof of Theorem 4.3 (max case)

Assume for any $\tilde{x} \in \Pi(x_0)$

$$\lim_{T \to \infty} \beta^T V(x_T) = 0.$$ 

BE implies

$$V(x_0) \geq F(x_0, x_1) + \beta V(x_1), \text{ all } x_1 \in \Gamma(x_0)$$

$$= F(x_0, x_1^*) + \beta V(x_1^*), \text{ some } x_1^* \in \Gamma(x_0)$$

Substituting $V(x_1)$:

$$V(x_0) \geq F(x_0, x_1) + \beta F(x_1, x_2) + \beta^2 V(x_2), \text{ all } x \in \Pi(x_0)$$

$$= F(x_0, x_1^*) + \beta F(x_1^*, x_2^*) + \beta^2 V(x_2^*), \text{ some } x^* \in \Pi(x_0)$$
Continue this way

\[ V(x_0) \geq \sum_{t=0}^{n} \beta^t F(x_t, x_{t+1}) + \beta^{n+1} V(x_{n+1}) \text{ for all } x \in \Pi(x_0) \]

\[ = \sum_{t=0}^{n} \beta^t F(x^*_t, x^*_{t+1}) + \beta^{n+1} V(x^*_{n+1}) \text{ for some } x^* \in \Pi(x_0) \]

Since \( \beta^T V(x_T) \to 0 \), taking the limit \( n \to \infty \) on both sides of both expressions we conclude that:

\[ V(x_0) \geq u(\tilde{x}) \text{ for all } \tilde{x} \in \Pi(x_0) \]

\[ V(x_0) = u(\tilde{x}^*) \text{ for some } \tilde{x}^* \in \Pi(x_0) \]
Bellman Equation as a Fixed Point

- define operator

\[ T(f)(x) = \max_{y \in \Gamma(x)} \{ F(x, y) + \beta f(y) \} \]

- \( V \) solution of BE \( \iff \) \( V \) fixed point of \( T \) [i.e. \( TV = V \)]

**Bounded Returns:**

- if \( \|F\| < B \) and \( F \) and \( \Gamma \) are continuos: \( T \) maps continuous bounded functions into continuous bounded functions

- bounded returns \( \Rightarrow \) \( T \) is a Contraction Mapping \( \Rightarrow \) unique fixed point

- many other bonuses
Needed Tools

- Basic Real Analysis (section 3.1):
  \{vector, metric, noSLP, complete\} spaces
  cauchy sequences
  closed, compact, bounded sets
- Contraction Mapping Theorem (section 3.2)
- Theorem of the Maximum: study of RHS of Bellman equation (equiva-
  lently of $T$) (section 3.3)