Recursive Methods
Outline Today’s Lecture

- “Anything goes”: Boldrin Montrucchio
- Global Stability: Liapunov functions
- Linear Dynamics
- Local Stability: Linear Approximation of Euler Equations
treat $X = [0, 1] \in \mathbb{R}$ case for simplicity

- take any $g(x) : [0, 1] \rightarrow [0, 1]$ that is twice continuously differentiable on $[0, 1]$
  $\Rightarrow g'(x)$ and $g''(x)$ exists and are bounded

- define
  $$W(x, y) = -\frac{1}{2}y^2 + yg(x) - \frac{L}{2}x^2$$

- Lemma: $W$ is strictly concave for large enough $L$
Proof

\[ W(x, y) = -\frac{1}{2}y^2 + yg(x) - \frac{L}{2}x^2 \]

\[ W_1 = yg'(x) - Lx \]
\[ W_2 = -y + g(x) \]

\[ W_{11} = yg''(x) - L \]
\[ W_{22} = -1 \]
\[ W_{12} = g'(x) \]

thus \( W_{22} < 0; \ W_{11} < 0 \) is satisfied if \( L \geq \max_x |g''(x)| \)

\[ W_{11}W_{22} - W_{12}W_{21} = -yg''(x) + L - g'(x)^2 > 0 \]
\[ \Rightarrow \ L > g'(x)^2 + yg''(x) \]

then \( L > \left[ \max_x |g'(x)| \right]^2 + \max_x |g''(x)| \) will do.
Decomposing $W$ (in a concave way)

- define $V(x) = W(x, g(x))$ and $F$ so that
  \[
  W(x, y) = F(x, y) + \beta V(y)
  \]
  that is $F(x, y) = W(x, y) - \beta V(y)$.

- Lemma: $V$ is strictly concave
  
  Proof: immediate since $W$ is concave and $X$ is convex. Computing the second derivative is useful anyway:
  \[
  V''(x) = g''(x)g(x) + g'(x)^2 - L
  \]
  since $g \in [0, 1]$ then clearly our bound on $L$ implies $V''(x) < 0$. 
Concavity of $F$

- Lemma: $F$ is concave for $\beta \in \left[0, \tilde{\beta}\right]$ for some $\tilde{\beta} > 0$

\[
F_{11}(x, y) = W_{11}(x, y) = yg''(x) - L \\
F_{12}(x, y) = W_{12}(x, y) = -1 \\
F_{22}(x, y) = W_{22} - \beta V_{22} = -1 - \beta \left[ gp''(x) g(x) + g'(x)^2 - L \right] \\
F_{11}F_{22} - F_{12}^2 > 0 \\
\Rightarrow (yg''(x) - L) \left(-1 - \beta \left[ gp''(x) g(x) + g'(x)^2 - L \right]\right) - g'(x)^2 > 0
\]
... concavity of \( F \)

- Let

\[
\eta_1 (\beta) = \min_{x,y} (-F_{22})
\]

\[
\eta_2 (\beta) = \min_{x,y} [F_{11}F_{22} - F_{12}^2]
\]

\[
\eta (\beta) = \min \{H_1 (\beta), H_2 (\beta)\} \geq 0
\]

- for \( \beta = 0 \) \( \eta (\beta) > 0 \). \( \eta \) is continuous (Theorem of the Maximum) \( \Rightarrow \) exists \( \tilde{\beta} > 0 \) such that \( H (\beta) \geq 0 \) for all \( \beta \in [0, \tilde{\beta}] \).
Monotonicity

- Use
  \[ W(x, y) = -\frac{1}{2}y^2 + yg(x) - \frac{L_1}{2}x^2 + L_2x \]
- \( L_2 \) does not affect second derivatives
- Claim: \( F \) is monotone for large enough \( L_2 \)