Auctions 4: Multiunit Auctions & Cremer-McLean Mechanism

$M$ units of the same object are offered for sale.

Each bidder has a set of (marginal values) $V^i = (V_1^i, V_2^i, \ldots V_M^i)$, the objects are substitutes, $V_k^i \geq V_{k+1}^i$.

Extreme cases: unit-demand, the same value for all objects.

- Types of auctions:
  - The discriminatory ("pay-your-bid");
  - Uniform-price;
  - Vickrey;
  - Multi-unit English;
7 Vickrey Auction

- Let $(b_1^i, b_2^i, \ldots, b_n^i)$ be the vector of bids submitted by $i$.

- Winners: $M$ highest bids.

- Payments: If player $i$ wins $m$ objects, then has to pay the sum of $m$ highest non-winning bids from the others.

  Or, price for each unit is: minimal value to have and win.

  E.g. to win 3d unit need to bid among $(M - 2)$ highest bids, $p = (M - 2)sd$ highest bid of the others.

- Weakly dominant to bid truthfully, $b_k^i = V_k^i$. 

Issues: Existence and description of equilibria, price series if sequential, efficiency, optimality, non-homogenous goods, complementarities,...
8 Interdependent valuations

8.1 Notation

$K$ objects; given $k = (k_1, \ldots, k_N)$, denote

$$V^k = (V_1^{k_1}, \ldots, V_N^{k_N}).$$

Winners circle at $s$, $\mathcal{I}^k(s)$, is the set of bidders with the highest value among $V^k$.

$k$ is admissible if $1 \leq k_i \leq K$ and

$$0 \leq \sum_{i=1}^{N} (k_i - 1) < K.$$

8.2 Single-crossing condition

MSC (single-crossing) For any admissible $k$, for all $x$ and any pair of players $\{i, j\} \subset \mathcal{I}^k(x)$,

$$\frac{\partial V_i^{k_i}(x)}{\partial x_i} > \frac{\partial V_j^{k_j}(x)}{\partial x_i}.$$

8.3 Efficiency: VCG mechanism (generalized Vickrey auction)

- Allocation rule: Efficient.

- Payments: Vickrey price that player $j$ pays for $k$th unit won:
\[ p^k_j = V^k_j(s^k_j, x_{-j}) = (M - k + 1) \text{th highest among } \{V^m_i(s^k_j, x_{-j})\}_{i \neq j}^{m=1..M}. \]

These are generically different across units and winners (unlike with private values).

9 Cremer & McLean Mechanism

- Multiple units. Single-crossing and non-independent values.
- Efficient, Extract all the surplus.

Discrete support: \( \mathcal{X}^i = \{0, \Delta, 2\Delta, \ldots, (t_i - 1)\Delta\} \), discrete single-crossing is assumed (no need if the values are private).

\( \Pi(x) \) is the joint probability of \( x \), \( \Pi_i = (\pi(x_{-i}|x_i)) \).

**Theorem:** In the above conditions and if \( \Pi \) has a full rank, there exists a mechanism in which truth-telling is an efficient ex post equilibrium and in which the seller extracts full surplus from the bidders.

Proof: Consider VCG mechanism \((Q^*, M^*)\). Define,

\[ U^*_i(x_i) = \sum_{x_{-i}} \pi(x_{-i}|x_i) [Q^*_i(x)V_i(x) - M^*_i(x)]. \]
This is the expected surplus of buyer $i$ in VCG mechanism. Define, $u_i^* = (U_i^*(x_i))_{x_i \in X^i}$.

There exists $c_i = (c_i(x_{-i}))_{x_{-i} \in X_{-i}}$, such that $\Pi_i c_i = u_i^*$. Equivalently,

$$\sum_{x_{-i}} \pi(x_{-i}|x_i)c_i(x_{-i}) = U_i^*(x_i).$$

Then, $CM$ mechanism $(Q^*, M^{CM}_i)$ is defined by

$$M^{CM}_i(x) = M_i^*(x) + c_i(x_{-i}).$$

Remarks:

- Private values (correlated), equiv. second price auction with additional payments.

- Negative payoffs sometimes, not ex post IR, payoffs arbitrarily large if the distribution converges to the independent one.
\[ p_j^k = V_j^k(s_j^k, x_{-j}) = \]

\((M - k + 1)\text{th highest}\)

among \(\{V_i^m(s_j^k, x_{-j})\}_{i \neq j}^{m=1..M}\).

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8.3 Efficiency: VCG mechanism (*generalized Vickrey auction*)

- Allocation rule: Efficient.

- Payments: Vickrey price that player $j$ pays for
  $k$th unit won:
• Ausubel;

• Dutch, descending uniform-price,

• ...

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