14.13 Economics and Psychology
(Lecture 5)

Xavier Gabaix

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1 Second order risk aversion for EU

• The agent takes the 50/50 gamble $\Pi + \sigma$, $\Pi - \sigma$ iff:

$$B(\Pi) = \frac{1}{2}u(x + \sigma + \Pi) + \frac{1}{2}u(x - \sigma + \Pi) \geq u(x)$$

i.e. $\Pi \geq \Pi^*$ where:

$$B(\Pi^*) = u(x)$$

• Assume that $u$ is twice differentiable and take a look at the Taylor expansion of the above equality for small $\sigma$.

$$B(\Pi) = u(x) + \frac{1}{2}u'(x)2\Pi + \frac{1}{4}u''(x)2\left[\sigma^2 + \Pi^2\right] + o\left(\sigma^2 + \Pi^2\right) = u(x)$$
then

\[ \Pi = \frac{\rho}{2} \left[ \sigma^2 + \Pi^2 \right] + o \left( \sigma^2 + \Pi^2 \right) \]

where \( \rho = -\frac{u''}{u'} \)

- To solve: \( \Pi = \frac{\rho}{2} \left[ \sigma^2 + \Pi^2 \right] \) for small \( \sigma \). Call \( \rho' = \rho/2 \).
• *Barbarian way*: Solve:

\[ \Pi^2 - \frac{1}{\rho'} \Pi + \sigma^2 = 0 \]

Exactly. Then take Taylor. One finds:

\[ \Pi = \rho' \sigma^2 = \frac{\rho}{2} \sigma^2 \]
Elegant way: $\Pi = \rho' \left[ \sigma^2 + \Pi^2 \right]$ for small $\sigma$.

- $\Pi$ will be small. Take a guess. If the expansion is $\Pi = k\sigma$, then we get:

$$k\sigma = \rho' \left[ \sigma^2 + k^2\sigma^2 \right]$$
$$k = \rho' \sigma \left[ 1 + k^2 \right]$$

contraction for $\sigma \to 0$, the RHS goes to 0 and the LHS is $k$. This guess doesn’t work.

- Let’s try instead $\Pi = k\sigma^2$. Then:

$$k\sigma^2 = \rho' \left[ \sigma^2 + k^2\sigma^4 \right]$$
$$= \rho'\sigma^2 + o \left( \sigma^2 \right)$$

$\Rightarrow k = \rho' + o(1)$ after dividing both side by $\sigma^2$
that works, with $k = \rho'$. Conclusion:

$$\Pi = \frac{\rho}{2}\sigma^2.$$ 

- Note this method is really useful when the equation to solve doesn’t have a closed form solution. For example, solve for small $\sigma$

$$\pi = \rho'(\sigma^2 + \pi^2 + \pi^7)$$

solution postulate $\Pi = k\sigma^2$, plug it back in the equation to solve, then take $\sigma \to 0$ and it works for $k = \rho'$

- The $\sigma^2$ indicates “second order” risk aversion.
2 First order risk aversion of PT

• Consider same gamble as for EU. Take the gamble iff $\Pi \geq \Pi^*$ where

$$\pi(.5)u(\Pi^* + \sigma) + \pi(.5)u(\Pi^* - \sigma) = 0$$

• We will show that in PT, as $\sigma \to 0$, the risk premium $\Pi$ is of the order of $\sigma$ when reference wealth $x = 0$. This is called the first order risk aversion.

• Let’s compute $\Pi$ for $u(x) = x^\alpha$ for $x \geq 0$ and $u(x) = -\lambda |x|^\alpha$ for $x \leq 0$. 

The premium $\Pi$ at $x = 0$ satisfies

$$0 = \pi \left( \frac{1}{2} \right) (\sigma + \Pi^*)^\alpha + \pi \left( \frac{1}{2} \right) (-\lambda) |\sigma + \Pi^*|^\alpha$$

cancel $\pi \left( \frac{1}{2} \right)$ and use the fact that $-\sigma + \Pi^* < 0$ to get

$$0 = (\sigma + \Pi^*)^\alpha - \lambda (\sigma - \Pi^*)^\alpha$$

$$\iff (\sigma + \Pi^*)^\alpha = \lambda (\sigma - \Pi^*)^\alpha$$

$$\iff \sigma + \Pi^* = \lambda^{1/\alpha} [\sigma - \Pi^*]$$

then

$$\Pi^* = \frac{1}{\lambda^\alpha - 1} \cdot \frac{1}{\sigma} = k \sigma$$

where $k$ is defined appropriately.
• Empirically:

\[
\lambda = 2, \quad \alpha \approx 1
\]

\[
k \approx \frac{2 - 1}{2 + 1} = \frac{1}{3}
\]

• Note that when \( \lambda = 1 \), the agent is risk neutral and the risk premium is 0.
2.1 Calibration 1

- Consider an EU agent with a constant elasticity of substitution, CES, utility, i.e. \( u(c) = \frac{c^{1-\gamma}}{1-\gamma} \).

- **Gamble 1**
  - $50,000 with probability 1/2
  - $100,000 with probability 1/2

- **Gamble 2.** $x$ for sure.

- Typical $x$ that makes people indifferent between the two gambles belongs to $(60k, 75k)$ (though some people are risk loving and ask for higher $x$).
• If $x = 65k$, what is $\gamma$

\[
.5 \ u(W + 50) + .5 \ u(W + 100) = u(W + x) \\
.5 \cdot W^{1-\gamma} \cdot 50^{1-\gamma} + .5 \cdot W^{1-\gamma} \cdot 100^{1-\gamma} = W^{1-\gamma} \cdot x^{1-\gamma} \\
5 \cdot 50^{1-\gamma} + .5 \cdot 100^{1-\gamma} = x^{1-\gamma}
\]

• Note the relation between $x$ and the elasticity of substitution $\gamma$:

\[
\begin{array}{cccccccccc}
x & 75k & 70k & 63k & 58k & 54k & 51.9k & 51.2k \\
\gamma & 0 & 1 & 3 & 5 & 10 & 20 & 30
\end{array}
\]

Right $\gamma$ seems to be between 1 and 10.

• Evidence on financial markets calls for $\gamma$ bigger than 10. This is the equity premium puzzle.
2.2 Calibration 2

- **Gamble 1**
  - $11 with probability 1/2
  - $-10 with probability 1/2

- **Gamble 2.** Get $0 for sure.

- If someone prefers Gamble 2, she or he satisfies

  \[ u(W) > \frac{1}{2}u(W + \Pi - \sigma) + \frac{1}{2}u(W + \Pi + \sigma) \].

  Here, $\Pi = .5$ and $\sigma = $10.5. We know that in EU

  \[ \Pi < \Pi^* = \frac{\rho}{2}\sigma^2 \]
And thus with CES utility $\rho = -\frac{u''(W)}{u'(W)} = -\frac{-\gamma W^{-\gamma-1}}{W-\gamma} = \frac{\gamma}{W}$

$$\Pi < \frac{\rho \sigma^2}{2} = \frac{\gamma}{2W} \sigma^2 \Leftrightarrow \frac{2W \Pi}{\sigma^2} < \gamma$$

forces large $\gamma$ as the wealth $W$ is larger than $10^5$ easily.

- Here:

  $$\gamma > \frac{2W \Pi}{\sigma^2} = \frac{2 \cdot 10^5 \cdot .5}{10.5^2} \approx 10^3$$

- Conclusion: very hard to calibrate the same model to large and small gambles using EU.
2.3 Calibration Conclusions

• What would a PT agent do? If $\alpha = 1$, $\lambda = 2$, in calibration 2 he won’t take gamble 1 as

$$\pi(.5)11^\alpha + \pi(.5)(-\lambda \cdot 10^\alpha) = -9\pi(.5) < 0$$

• In PT we have $\Pi^* = k:\sigma$. For $W = 10^4$, $\gamma = 2$, and $\sigma = 0.5$ the risk premium is $\Pi^* = k:\sigma = \frac{1}{3} \cdot .5 \approx .2$ while in EU $\Pi^* = \frac{\gamma}{2W}\sigma^2 \approx .00002$

• If we want to fit an EU parameter $\gamma$ to a PT agent we get

$$\Pi^{PT}(\sigma) = \Pi^{EU}(\sigma)$$

$$k:\sigma = \frac{\gamma}{2W}\sigma^2$$
then
\[ \hat{\gamma} = \frac{2kW}{\sigma} \]
and this explodes as \( \sigma \to 0 \).
• If someone is averse to 50-50 lose $100/gain $g$ for all wealth levels then he or she will turn down 50-50 lose $L$-gain $G$ in the table

• Guess:

<table>
<thead>
<tr>
<th>$L \backslash g$</th>
<th>$101$</th>
<th>$105$</th>
<th>$110$</th>
<th>$125$</th>
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<tbody>
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<td>$400$</td>
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</tbody>
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\[
\begin{array}{lcccc}
L \setminus g & $101$ & $105$ & $110$ & $125$ \\
$400$ & $400$ & $420$ & $550$ & $1,250$ \\
$800$ & $800$ & $1,050$ & $2,090$ & $\infty$ \\
$1000$ & $1,010$ & $1,570$ & $\infty$ & $\infty$ \\
$2000$ & $2,320$ & $\infty$ & $\infty$ & $\infty$ \\
$10,000$ & $\infty$ & $\infty$ & $\infty$ & $\infty$
\end{array}
\]

cf paper by Matt Rabin
2.4 What does it mean?

- EU is still good for modelling.

- Even behavioral economists stick to it when they are not interested in risk taking behavior, but in fairness for example.

- The reason is that EU is nice, simple, and parsimonious.


3 Two extensions of PT

• Both outcomes, \( x \) and \( y \), are positive, \( 0 < y < x \). Then,

\[
V = v(y) + \pi(p)(v(x) - v(y)).
\]

Why not \( V = \pi(p)v(x) + \pi(1-p)v(y) \)? Because it becomes self-contradictory when \( x = y \) and we stick to K-T calibration that puts \( \pi(.5) < .5 \).
Continuous gambles, distribution $f(x)$

EU gives:

$$ V = \int_{-\infty}^{+\infty} u(x) f(x) \, dx $$

PT gives:

$$ V = \int_{0}^{+\infty} u(x) f(x) \pi'(P(g \geq x)) \, dx $$

$$ + \int_{-\infty}^{0} u(x) f(x) \pi'(P(g \leq x)) \, dx $$