Coalitional Games

A coalitional (or cooperative) game is a model that focuses on the behavior of groups of players. The strategic interaction is not explicitly modeled as in the case of non-cooperative games.

- $N$: finite set of players
- a coalition is any group of players, $S \subseteq N$ ($N$ is the grand coalition)
- $v(S) \geq 0$: worth of coalition $S$
- $S$ can divide $v(S)$ among its members; $S$ may implement any payoffs $(x_i)_{i \in S}$ with $\sum_{i \in S} x_i = v(S)$ (no externalities)
- outcome: a partition $(S_k)_{k=1,\ldots,\bar{k}}$ of $N$ and an allocation $(x_i)_{i \in N}$ specifying the division of the worth of each $S_k$ among its members,

$$S_j \cap S_k = \emptyset, \forall j \neq k \quad \& \quad \bigcup_{k=1}^{\bar{k}} S_k = N$$

$$\sum_{i \in S_k} x_i = v(S_k), \forall k \in \{1, \ldots, \bar{k}\}$$
Examples

A majority game

- Three parties (players 1, 2, and 3) can share a unit of total surplus.
- Any majority—coalition of 2 or 3 parties—may control the allocation of output.
- Output is shared among the members of the winning coalition.

\[
\begin{align*}
  v(\{1\}) &= v(\{2\}) = v(\{3\}) = 0 \\
  v(\{1, 2\}) &= v(\{1, 3\}) = v(\{2, 3\}) = v(\{1, 2, 3\}) = 1
\end{align*}
\]

Firm and workers

- A firm, player 0, may hire from the pool of workers \{1, 2, \ldots, n\}.
- Profit from hiring \(k\) workers is \(f(k)\).

\[
v(S) = \begin{cases} 
  f(|S| - 1) & \text{if } 0 \in S \\
  0 & \text{otherwise}
\end{cases}
\]
The Core

Suppose that it is efficient for the grand coalition to form:

\[ v(N) \geq \sum_{k=1}^{\bar{k}} v(S_k) \text{ for every partition } (S_k)_{k=1,...,\bar{k}} \text{ of } N. \]

Which allocations \((x_i)_{i \in N}\) can the grand coalition choose? No coalition \(S\) should want to break away from \((x_i)_{i \in N}\) and implement a division of \(v(S)\) that all its members prefer to \((x_i)_{i \in N}\).

For an allocation \((x_i)_{i \in N}\), use notation \(x_S = \sum_{i \in S} x_i\). Allocation \((x_i)_{i \in N}\) is feasible for the grand coalition if \(x_N = v(N)\).

**Definition 1**

Coalition \(S\) can block the allocation \((x_i)_{i \in N}\) if \(x_S < v(S)\). An allocation is in the core of the game if (1) it is feasible for the grand coalition; and (2) it cannot be blocked by any coalition. \(C\) denotes the set of core allocations,

\[ C = \{(x_i)_{i \in N} | x_N = v(N) \& x_S \geq v(S), \forall S \subseteq N\}. \]
Examples

- Two players split $1, with outside options $p$ and $q$

  \[ v(\{1\}) = p, \quad v(\{2\}) = q, \quad v(\{1, 2\}) = 1 \]

  \[ C = \{(x_1, x_2) | x_1 + x_2 = 1, \ x_1 \geq p, \ x_2 \geq q\} \]

  What happens for $p = q = 0$? What if $p + q > 1$?

- The majority game

  \[ v(\{1\}) = v(\{2\}) = v(\{3\}) = 0 \]

  \[ v(\{1, 2\}) = v(\{1, 3\}) = v(\{2, 3\}) = v(\{1, 2, 3\}) = 1 \]

  \[ C = ? \]

- A set $A$ of 1000 sellers interacts with a set $B$ of 1001 buyers in a market for an indivisible good. Each seller supplies one unit of the good and has reservation value 0. Every buyer demands a single unit and has reservation price 1.

  \[ v(S) = \min(|S \cap A|, |S \cap B|) \]

  \[ C = ? \]
Balancedness

Which games have nonempty core?

A vector \((\lambda_S \geq 0)_{S \subseteq N}\) is balanced if

\[
\sum_{\{S \subseteq N | i \in S\}} \lambda_S = 1, \forall i \in N.
\]

A payoff function \(v\) is balanced if

\[
\sum_{S \subseteq N} \lambda_S v(S) \leq v(N) \text{ for every balanced } \lambda.
\]

Interpretation: each player has a unit of time, which can be distributed among his coalitions. If each member of coalition \(S\) is active in \(S\) for \(\lambda_S\) time, a payoff of \(\lambda_S v(S)\) is generated. A game is balanced if there is no allocation of time across coalitions that yields a total value > \(v(N)\).
Balancedness is **Necessary** for a Nonempty Core

Suppose that $C \neq \emptyset$ and consider $x \in C$. If $(\lambda_S)_{S \subseteq N}$ is balanced, then

$$\sum_{S \subseteq N} \lambda_S v(S) \leq \sum_{S \subseteq N} \lambda_S x_S = \sum_{i \in N} x_i \sum_{S \ni i} \lambda_S = \sum_{i \in N} x_i = v(N).$$

Hence $v$ is balanced.

Balancedness turns out to be also a **sufficient** condition for the non-emptiness of the core...
Nonempty Core

Theorem 1 (Bondareva 1963; Shapley 1967)

A coalitional game has non-empty core iff it is balanced.
Proof

Consider the linear program

\[ X := \min \sum_{i \in N} x_i \]

s.t. \[ \sum_{i \in S} x_i \geq v(S), \forall S \subseteq N. \]

\( C \neq \emptyset \iff X \leq v(N) \) (1)

Dual program

\[ Y := \max \sum_{S \subseteq N} \lambda_S v(S) \]

s.t. \( \lambda_S \geq 0, \forall S \subseteq N \& \sum_{S \ni i} \lambda_S = 1, \forall i \in N. \)

\( v \) is balanced \( \iff Y \leq v(N) \) (2)

The primal linear program has an optimal solution. By the duality theorem of linear programming, \( X = Y \) (3).

(1)-(3): \( C \neq \emptyset \iff v \) is balanced
Simple Sufficient Condition for Nonempty Cores

Definition 2

A game $v$ is convex if for any pair of coalitions $S$ and $T$,

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T).$$

Convexity implies that the marginal contribution of a player $i$ to a coalition increases as the coalition expands,

$$S \subset T \& i \notin T \implies v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S).$$

Indeed, if $v$ is convex then

$$v((S \cup \{i\}) \cup T) + v((S \cup \{i\}) \cap T) \geq v(S \cup \{i\}) + v(T),$$

which can be rewritten as

$$v(T \cup \{i\}) - v(T) \geq v(S \cup \{i\}) - v(S).$$
Convex Games Have Nonempty Cores

**Theorem 2**

*Every convex game has a non-empty core.*

Define the allocation $x$ with $x_i = v\{1, \ldots, i\} - v\{1, \ldots, i - 1\}$. Prove that $x \in C$. For all $i_1 < i_2 < \cdots < i_k$,

$$
\sum_{j=1}^{k} x_{i_j} = \sum_{j=1}^{k} v\{1, \ldots, i_j - 1, i_j\} - v\{1, \ldots, i_j - 1\} \\
\geq \sum_{j=1}^{k} v\{i_1, \ldots, i_{j-1}, i_j\} - v\{i_1, \ldots, i_{j-1}\} \\
= v\{i_1, i_2, \ldots, i_k\},
$$

where the inequality follows from $\{i_1, \ldots, i_{j-1}\} \subseteq \{1, \ldots, i_j - 1\}$ and $v$’s convexity.
Core Tâtonnement

Consider a game \( \nu \) with \( C \neq \emptyset \).

\[ e(S; x) = \nu(S) - x_S: \] excess of coalition \( S \) at allocation \( x \)

\[ D(x) \subseteq 2^N: \] most discontent coalitions at \( x \),

\[ D(x) = \arg \max_{S \in N} w(S)e(S; x) \]

where \( w : 2^N \rightarrow (0, \infty) \) describes coalitions’ relative ability of expressing discontent and threatening to block

For any feasible allocation \( x^0 \), consider the following recursive process. For \( t = 1, 2, \ldots \)

\[ \text{if } x^{t-1} \in C, \text{ then } x^t = x^{t-1}; \]

\[ \text{otherwise, one coalition } S^{t-1} \in D(x^{t-1}) \text{ most discontent with } x^{t-1} \text{ is chosen and } e(S^{t-1}; x^{t-1}) \text{ is transferred symmetrically from } N \setminus S^{t-1} \text{ to } S^{t-1}, \]

\[ x^t_i = \begin{cases} 
    x_i^{t-1} + \frac{e(S^{t-1}; x^{t-1})}{|S^{t-1}|} & \text{if } i \in S^{t-1} \\
    x_i^{t-1} - \frac{e(S^{t-1}; x^{t-1})}{|N \setminus S^{t-1}|} & \text{if } i \in N \setminus S^{t-1}.
\end{cases} \]
The sequence \((x^t)\) converges to a core allocation.

For intuition, view allocations \(x\) as elements of \(\mathbb{R}^N\).

- \((x^t)\) is confined to the hyperplane \(\{x|x_N = v(N)\}\).
- Assume that \((x^t)\) does not enter \(C\).
- At each step \(t\), the reallocation is done such that \(x^{t+1}\) is the projection of \(x^t\) on the hyperplane \(F_{S^t}\), where \(F_S = \{x|x_S = v(S) \& x_N = v(N)\}\).
- Distance from \(x^t\) to \(F_{S^t}\) is proportional to \(e(S^t; x^t)\).
- For any fixed \(c \in C\), since \(x^t\) and \(c\) are on different sides of the hyperplane \(F_{S^t}\) and the line \(x^t x^{t+1}\) is perpendicular to \(F_{S^t}\), we have \(\hat{x^tx^{t+1}}c > \pi/2\) and \(d(x^t, c) \geq d(x^{t+1}, c)\) for all \(t \geq 0\).
- \(l_c := \lim_{t \to \infty} d(x^t, c)\)
Continuation of Proof Sketch

- For any limit point \( x \) of \((x_t)\), there exists a subsequence of \((x_t)\) converging to \( x \) and a coalition \( S \) such that \( S^t = S \) along the subsequence.

- The projection of the subsequence on \( F_S \) converges to the projection \( y \) of \( x \) on \( S \) \( \Rightarrow \) \( y \) is also a limit point.

- If \( x \not\in F_S \) (\( x \neq y \)), then for any \( c \in C \) the segment \( xc \) is longer than \( yc \) because \( \overline{xy}c > \pi/2 \). This contradicts \( d(x, c) = d(y, c) = l_c \).

- Therefore, \( x \in F_S \) and \( e(S; x) = 0 \). Then \( x \in C \) since, by continuity, \( S \) is one of the most discontent coalitions under \( x \) \( \Rightarrow \) \( l_x = 0 \).

- Any other limit point \( z \) satisfies \( d(z, x) = l_x = 0 \), so \( z = x \).

- \((x_t)\) converges to \( x \in C \).
Singleton Solution Concepts

Two players split $1, with outside options \( p \) and \( q \)

\[ v(\{1\}) = p, \ v(\{2\}) = q, \ v(\{1, 2\}) = 1 \]

\[ C = \{ (x_1, x_2) | x_1 + x_2 = 1, \ x_1 \geq p, \ x_2 \geq q \} \]

What happens for \( p = q = 0 \)? What if \( p + q > 1 \)?

The core may be empty or quite large, which compromises its role as a predictive theory. Ideally, select a unique outcome for every cooperative game.

A value for cooperative games is a function from the space of games \((N, v)\) to feasible allocations \( x \ (x_N = v(N)) \).
The Shapley Value

Shapley (1953) proposed a solution with many economically desirable and mathematically elegant properties.

Definition 3

The Shapley value of a game with worth function \( v \) is given by

\[
\phi_i(v) = \sum_{S \subseteq \mathbb{N} \setminus \{i\}} \frac{|S|!(|\mathbb{N}| - |S| - 1)!}{|\mathbb{N}|!} (v(S \cup \{i\}) - v(S)).
\]

Interpretation: players are randomly ordered in a line, all orders being equally likely. \( \phi_i(v) \) represents the expected value of player \( i \)'s contribution to the coalition formed by the players preceding him in line.

Why do values sum to \( v(N) \)?

What's the Shapley value in the divide the dollar game?

Proposition 2 ⇒ for convex games \( v \), \( \phi(v) \) is a convex combination of core allocations. Since \( C \) is convex, \( \phi(v) \in C \). Not true in general.
Axioms

What is special about the Shapley value?

**Axiom 1 (Symmetry)**

*Players i and j are interchangeable in v if* \( v(S \cup \{i\}) = v(S \cup \{j\}) \) *for all S disjoint from \( \{i, j\} \). If i and j are interchangeable in v then* \( \phi_i(v) = \phi_j(v) \).

**Axiom 2 (Dummy Player)**

*Player i is a dummy in v if* \( v(S \cup \{i\}) = v(S) \) *for all S. If i is a dummy in v then* \( \phi_i(v) = 0 \).

**Axiom 3 (Additivity)**

*For any two games v and w, we have* \( \phi(v + w) = \phi(v) + \phi(w) \).

**Theorem 4**

*A value satisfies the three axioms iff it is the Shapley value.*
Proof of “If” Part

The only axiom not checked immediately is symmetry. Suppose that \( i \) and \( j \) are interchangeable. Then

\[
\varphi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} (v(S \cup \{i\}) - v(S))
\]

\[
= \sum_{S \subseteq N \setminus \{i,j\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} (v(S \cup \{i\}) - v(S))
\]

\[
+ \sum_{S \subseteq N \setminus \{i,j\}} \frac{(|S| + 1)!(|N| - (|S| + 1) - 1)!}{|N|!} (v(S \cup \{i,j\}) - v(S \cup \{j\}))
\]

\[
= \sum_{S \subseteq N \setminus \{i,j\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} (v(S \cup \{j\}) - v(S))
\]

\[
+ \sum_{S \subseteq N \setminus \{i,j\}} \frac{(|S| + 1)!(|N| - (|S| + 1) - 1)!}{|N|!} (v(S \cup \{i,j\}) - v(S \cup \{i\}))
\]

\[
= \varphi_j(v).
\]
Proof of “Only If” Part

Suppose that \( \psi \) satisfies the three axioms. We argue that \( \psi = \varphi \).

For any non-empty coalition \( T \), define the game

\[
\nu^T(S) = \begin{cases} 
1 & \text{if } S \supseteq T \\
0 & \text{otherwise} 
\end{cases}.
\]

Fix \( a \in \mathbb{R} \). By the symmetry axiom, \( \psi_i(\av^T) = \psi_j(\av^T) \) for all \( i, j \in T \). By the dummy player axiom, \( \psi_i(\av^T) = 0 \) for all \( i \not\in T \). Hence

\[
\psi_i(\av^T) = \begin{cases} 
\frac{a}{|T|} & \text{if } i \in T \\
0 & \text{otherwise} 
\end{cases},
\]

so \( \psi(\av^T) = \varphi(\av^T) \).
Proof of “Only If” Part

The \((2^{|N|} - 1)\) games \(v^T\) span the linear space of all games. If we view games as \((2^{|N|} - 1)\)-dimensional vectors, it is sufficient to show that the vectors corresponding to the games \((v^T)\) are linearly independent.

For a contradiction, suppose that \(\sum_{T \subseteq N} \alpha^T v^T = 0\) with not all \(\alpha\)'s equal to zero. Let \(S\) be a set with minimal cardinality satisfying \(\alpha^S \neq 0\). Then \(\sum_{T \subseteq N} \alpha^T v^T (S) = \alpha^S \neq 0\), a contradiction.

Thus any \(v\) can be written as \(v = \sum_{T \subseteq N} \alpha^T v^T\). The additivity of \(\psi\) and \(\varphi\) imply

\[
\psi(v) = \psi \left( \sum_{T \subseteq N} \alpha^T v^T \right) = \sum_{T \subseteq N} \psi(\alpha^T v^T) = \sum_{T \subseteq N} \varphi(\alpha^T v^T) = \varphi \left( \sum_{T \subseteq N} \alpha^T v^T \right) = \varphi(v).
\]
An Alternative Characterization

*Equity* requirement: for any pair of players, the amounts that each player gains or loses from the other’s withdrawal from the game are equal. For a game \((N, v)\), we denote by \(v|M\) its restriction to the players in \(M\).

**Definition 4**

A value \(\psi\) has balanced contributions if for every game \((N, v)\) we have

\[
\psi_i(v|N) - \psi_i(v|N \setminus \{j\}) = \psi_j(v|N) - \psi_j(v|N \setminus \{i\}), \quad \forall i, j \in N.
\]

**Theorem 5**

The unique value that has balanced contributions is the Shapley value.
Proof

At most one value has balanced contributions.

- For a contradiction, let $\varphi'$ and $\varphi''$ be two different such values.
- Let $(N, \nu)$ be a game with minimal $|N|$ for which the two values yield different outcomes.
- Then for all $i, j \in N$, $\varphi'_i(\nu|N \setminus \{j\}) = \varphi''_i(\nu|N \setminus \{j\})$ and $\varphi'_j(\nu|N \setminus \{i\}) = \varphi''_j(\nu|N \setminus \{i\})$, along with the balancedness of $\varphi'$ and $\varphi''$, imply $\varphi'_i(\nu|N) - \varphi''_i(\nu|N) = \varphi'_j(\nu|N) - \varphi''_j(\nu|N)$.
- Since $\sum_{i \in N}(\varphi'_i(\nu|N) - \varphi''_i(\nu|N)) = 0$, we obtain $\varphi'_i(\nu|N) - \varphi''_i(\nu|N) = 0, \forall i \in N$, or $\varphi'(\nu|N) = \varphi''(\nu|N)$, a contradiction.

Next argue that the Shapley value has balanced contributions.

- The Shapley value $\varphi$ is a linear function of the game, so the set of games for which $\varphi$ satisfies balanced contributions is closed under linear combinations.
- Since any game is a linear combination of games $\nu^T$, it is sufficient to show that these games satisfy balanced contributions. . .
The Bargaining Problem

The *non-cooperative approach* involves explicitly modeling the bargaining process as an extensive form game (e.g., Rubinstein’s (1982) alternating offer bargaining model).

The *axiomatic approach* abstracts away from the details of the bargaining process. Determine directly “reasonable” or “natural” properties that outcomes should satisfy.

What are “reasonable” axioms?

- Consider a situation where two players must split $1. If no agreement is reached, then the players receive nothing.
- If preferences over monetary prizes are identical, then we expect that each player obtains 50 cents.
- Desirable properties: efficiency and symmetry of the allocation for identical preferences.
Nash Bargaining Solution

A bargaining problem is a pair \((U, d)\) where \(U \subset \mathbb{R}^2\) and \(d \in U\).

- \(U\) is convex and compact
- there exists some \(u \in U\) such that \(u > d\)

Denote the set of all possible bargaining problems by \(\mathcal{B}\). A bargaining solution is a function \(f: \mathcal{B} \to \mathbb{R}^2\) with \(f(U, d) \in U\).

**Definition 5**

The Nash (1950) bargaining solution \(f^N\) is defined by

\[
\{f^N(U, d)\} = \arg \max_{u \in U, u \geq d} (u_1 - d_1)(u_2 - d_2).
\]

Given the assumptions on \((U, d)\), the solution to the optimization problem exists and is unique.
Axiom 4 (Pareto Efficiency)

A bargaining solution $f$ is Pareto efficient if for any bargaining problem $(U, d)$, there does not exist $(u_1, u_2) \in U$ such that $u_1 \geq f_1(U, d)$ and $u_2 \geq f_2(U, d)$, with at least one strict inequality.

Axiom 5 (Symmetry)

A bargaining solution $f$ is symmetric if for any symmetric bargaining problem $(U, d)$ ($(u_1, u_2) \in U$ if and only if $(u_2, u_1) \in U$ and $d_1 = d_2$), we have $f_1(U, d) = f_2(U, d)$. 
Axioms

Axiom 6 (Invariance to Linear Transformations)

A bargaining solution $f$ is invariant if for any bargaining problem $(U, d)$ and all $\alpha_i \in (0, \infty), \beta_i \in \mathbb{R}$ ($i = 1, 2$), if we consider the bargaining problem $(U', d')$ with

\[
U' = \{(\alpha_1 u_1 + \beta_1, \alpha_2 u_2 + \beta_2) | (u_1, u_2) \in U\}
\]
\[
d' = (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2)
\]

then $f_i(U', d') = \alpha_i f_i(U, d) + \beta_i$ for $i = 1, 2$.

Axiom 7 (Independence of Irrelevant Alternatives)

A bargaining solution $f$ is independent if for any two bargaining problems $(U, d)$ and $(U', d)$ with $U' \subseteq U$ and $f(U, d) \in U'$, we have

$f(U', d) = f(U, d)$. 
Theorem 6

\( f^N \) is the unique bargaining solution that satisfies the four axioms.

Check that \( f^N \) satisfies the axioms.

1. **Pareto efficiency**: follows from the fact that \((u_1 - d_1)(u_2 - d_2)\) is increasing in \(u_1\) and \(u_2\).

2. **Symmetry**: if \((U, d)\) is a symmetric bargaining problem then \((f_2^N(U, d), f_1^N(U, d)) \in U\) also solves the optimization problem. By the uniqueness of the optimal solution, \(f_1^N(U, d) = f_2^N(U, d)\).

3. **Independence of irrelevant alternatives**: if \(f^N(U, d) \in U' \subseteq U\). The value of the objective function for \((U', d)\) cannot exceed that for \((U, d)\). Since \(f^N(U, d) \in U'\), the two values must be equal, and by the uniqueness of the optimal solution, \(f^N(U, d) = f^N(U', d)\).

4. **Invariance to linear transformations**: \(f^N(U', d')\) is an optimal solution for

\[
\max \{ (u'_1, u'_2) | u'_1 = \alpha_1 u_1 + \beta_1, u'_2 = \alpha_2 u_2 + \beta_2, (u_1, u_2) \in U \} \]

\[ (u'_1 - \alpha_1 d_1 - \beta_1)(u'_2 - \alpha_2 d_2 - \beta_2) \ldots \]
Proof

Show that for any $f$ that satisfies the axioms, $f(U, d) = f^N(U, d), \forall (U, d)$.

Fix a bargaining problem $(U, d)$ and let $z = f^N(U, d)$. There exists $\alpha_i > 0, \beta_i$ such that the transformation $u_i \mapsto \alpha_i u_i + \beta_i$ takes $d_i$ to 0 and $z_i$ to 1/2. Define

$$U' = \{(\alpha_1 u_1 + \beta_1, \alpha_2 u_2 + \beta_2) | (u_1, u_2) \in U\}.$$ 

Since $f$ and $f^N$ satisfy the invariance to linear transformations axiom, $f(U, d) = f^N(U, d)$ iff $f(U', 0) = f^N(U', 0) = (1/2, 1/2)$. It suffices to prove $f(U', 0) = (1/2, 1/2)$. 

Proof

The line \( \{(u_1, u_2) | u_1 + u_2 = 1 \} \) is tangent to the hyperbola \( \{(u_1, u_2) | u_1 u_2 = 1/4 \} \) at the point \((1/2, 1/2)\). Given that \( f^N(U', 0) = (1/2, 1/2) \), argue that \( u_1 + u_2 \leq 1 \) for all \( u \in U' \).

Since \( U' \) is bounded, we can find a rectangle \( U'' \) with one side along the line \( u_1 + u_2 = 1 \), symmetric with respect to the line \( u_1 = u_2 \), such that \( U' \subseteq U'' \) and \((1/2, 1/2)\) is on the boundary of \( U'' \). Since \( f \) is efficient and symmetric, it must be that \( f(U'', 0) = (1/2, 1/2) \).

\( f \) satisfies independence of irrelevant alternatives, so \( f(U'', 0) = (1/2, 1/2) \in U' \subseteq U'' \Rightarrow f(U', 0) = (1/2, 1/2) \)
Bargaining with Alternating Offers

- players $i = 1, 2; j = 3 - i$
- set of feasible utility pairs

$$U = \{(u_1, u_2) \in [0, \infty)^2 | u_2 \leq g_2(u_1)\}$$

- $g_2$ s. decreasing, concave, $g_2(0) > 0$
- disagreement point $d = (0, 0)$
- $\delta_i$: discount factor of player $i$
- at every time $t = 0, 1, \ldots$, player $i(t)$ proposes an alternative $u = (u_1, u_2) \in U$ to player $j(t) = 3 - i(t)$

$$i(t) = \begin{cases} 
1 & \text{for } t \text{ even} \\
2 & \text{for } t \text{ odd} 
\end{cases}$$

- if $j(t)$ accepts the offer, game ends yielding payoffs $(\delta_1^t u_1, \delta_2^t u_2)$
- otherwise, game proceeds to period $t + 1$
Subgame perfect equilibrium

Define $g_1 = g_2^{-1}$. Graphs of $g_2$ and $g_1^{-1}$: Pareto-frontier of $U$

Let $(m_1, m_2)$ be the unique solution to the following system of equations

\[ m_1 = \delta_1 g_1 (m_2) \]
\[ m_2 = \delta_2 g_2 (m_1). \]

$(m_1, m_2)$ is the intersection of the graphs of $\delta_2 g_2$ and $(\delta_1 g_1)^{-1}$.

Subgame perfect equilibrium in “stationary” strategies: in any period where player $i$ has to make an offer to $j$, he offers $u$ with $u_j = m_j$ and $u_i = g_i(m_j)$, and $j$ accepts only offers $u$ with $u_j \geq m_j$. 

Nash Bargaining

Assume $g_2$ is decreasing, s. concave and continuously differentiable.

*Nash bargaining solution:*

$$\{u^*\} = \arg \max_{u \in U} u_1 u_2.$$  

**Theorem 7 (Binmore, Rubinstein and Wolinsky 1985)**  
Suppose that $\delta_1 = \delta_2 =: \delta$ in the alternating bargaining model. Then the unique SPE payoffs converge to the Nash bargaining solution as $\delta \to 1$. 

$$m_1 g_2 (m_1) = m_2 g_1 (m_2)$$

$(m_1, g_2 (m_1))$ and $(g_1 (m_2), m_2)$ belong to the intersection of $g_2$’s graph with the same hyperbola, which approaches the hyperbola tangent to the boundary of $U$ (at $u^*$) as $\delta \to 1$. 
14.16 Strategy and Information
Spring 2016

For information about citing these materials or our Terms of Use, visit: https://ocw.mit.edu/terms.