14.30 PROBLEM SET 9 SUGGESTED ANSWERS

TA: Tonja Bowen Bishop

Problem 1
For parts (a)-(g), please refer to the relevant sections in the handouts/textbooks.

(h) False. With the knowledge of the distribution of a statistic under the null hypothesis, we can calculate only $\alpha$, but not $\beta$.

(i) False. In particular, there are hypothesis tests for which one can form a GLRT where no optimal test exists, so GLRT could not be optimal in that case.

(j) Yes. Hypothesis tests are typically constructed by using a test statistic $T$. The null hypothesis is rejected if $T$ lies in some interval or if $T$ lies outside of some interval. The interval is chosen to make the test have a desired significance level. This procedure is in essence the same as the construction of the confidence interval.

(k) Recall that $\alpha$ and $\beta$ are calculated under different distributions (under the null and under the alternative hypothesis). There is no reason that it always must be $\alpha + \beta = 1$, although it is possible under some special circumstances. Also, in general, it is not possible $\alpha = \beta = 0$, again in some cases we might have $\alpha = 0$ or $\beta = 0$, but in many cases neither is possible.

NOTE: We might have some cases that $\alpha = \beta = 0$ under very special and unrealistic situation (can you think of an example of such case ?), and in that case, the hypothesis test becomes trivial and uninteresting. (why ?)

Problem 2
(a) The probability of committing a Type I error is calculated as follows:

$$P(\text{Type I error}) = P(\text{reject } H_0| H_0 \text{ is true}) = P(Y \geq 3.20| \lambda = 1) = \int_{3.20}^{\infty} e^{-\frac{y}{\lambda}} dy = 0.04$$

(b) The probability of committing a Type II error when $\lambda = 4/3$ is calculated as follows:
\[ P(\text{Type II error}) = P(\text{don’t reject } H_0|H_1 \text{ is true}) \]
\[ = P(Y \leq 3.20|\lambda = \frac{4}{3}) \]
\[ = \int_0^{3.20} \frac{3}{4} e^{-3y/4} dy \]
\[ = \int_0^{2.4} e^{-u} du \text{ (change in variable: } u = \frac{3}{4}y) \]
\[ = 0.91 \]

**Problem 3**

(a) i) We first find formulas for \( \alpha \) and \( \beta \) in terms of \( k \):

\[ \alpha = P(\text{Type I error}) = P(\text{reject } H_0|H_0 \text{ is true}) \]
\[ = \int_0^k f_0(x) dx = \int_0^k 2x dx \]
\[ = k^2 \]

\[ \beta = P(\text{Type II error}) = P(\text{don’t reject } H_0|H_A \text{ is true}) \]
\[ = \int_k^1 f_A(x) dx = \int_k^1 (2 - 2x) dx \]
\[ = k^2 - 2k + 1 \]

Note that we were able to write down expressions for \( \alpha \) and \( \beta \) because with only 1 observation \( x \), we knew that \( x \) had to be our statistic, and furthermore our test would be of the form “reject \( H_0 \) for \( x < k \)” because we are more likely to have observed a small \( x \) if \( H_A \) is true.

\[ \min_k (\alpha + \beta) = \min_k (2k^2 - 2k + 1) \]

\[ \frac{\partial (2k^2 - 2k + 1)}{\partial k} = 4k - 2 = 0 \]

So \( \alpha + \beta \) is minimized at \( k = 1/2 \).

ii) \( \alpha + \beta = \frac{1}{4} + \frac{1}{4} = \frac{1}{2} \).

iii) With \( k = 1/2 \), the testing procedure is then to reject the null for \( x < k = 1/2 \). So we do not reject the null for \( x = 0.6 \).

(b) i) Find \( k \) such that

\[ \int_0^k f_0(x) dx = \int_0^k 2x dx = k^2 = 0.1 \text{ [why 0.1, when the restriction is } \alpha \leq 0.10?] \]

This implies \( k = \sqrt{0.1} \).
ii) Then

\[ \beta = \int_{\sqrt{0.1}}^{1} (2 - 2x) dx \]

\[ = [2x - x^2]_{\sqrt{0.1}}^{1} \]

\[ = 1.1 - 2\sqrt{0.1} \]

iii) For \( x = 0.4 \), we do not reject because

\[ x = 0.4 > k = \sqrt{0.1} \]

(c) With ten observations, you could use, for instance

\[ T(X) = \overline{X} \]

and reject the if \( \overline{X} < k \) (for appropriate \( k \)).

**Problem 4**

We have two simple hypotheses, so we should be able to derive an optimal test statistic using the Neyman-Pearson lemma. The lemma tells us that the best test can be achieved by constructing the test statistic

\[ T = \frac{f_1(x_1, \ldots, x_n | \mu = 1)}{f_0(x_1, \ldots, x_n | \mu = 0)} \]

and then rejecting the null \( (\mu = 0) \) whenever \( T > k \), where \( k \) is the critical value such that the probability of type I error is equal to \( \alpha = 0.005 \). We know that the likelihood function is

\[ f(x_1, \ldots, x_n | \mu) = \prod_{i=1}^{n} f(x_i | \mu) \]

\[ = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x_i - \mu)^2}{2}} \]

\[ = (2\pi)^{-n/2} e^{-\frac{1}{2} \sum (x_i - \mu)^2} \]

Then we have

\[ T = \frac{(2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2} \sum (x_i - 1)^2}}{(2\pi)^{-\frac{n}{2}} e^{-\frac{1}{2} \sum x_i^2}} \]

\[ = e^{-\frac{1}{2} \sum (-2x_i + 1)} \]

\[ = e^{n\overline{X} - \frac{n}{2}} \]

Thus we should reject the null hypothesis if \( e^{n\overline{X} - \frac{n}{2}} > k \), or equivalently, reject the null if \( \overline{X} > \frac{2\ln(k) + n}{2n} \). Thus we have a test of the form "reject if \( \overline{X} > c \)" for some \( c(n) \), and we know how to determine the proper value of \( c \) for our desired confidence level. We want \( \Pr(\overline{X} > c | \mu = 0) = \alpha = 0.005 \),
so we use

\[
0.005 = \Pr(\bar{X} > c \mid \mu = 0) = \Pr\left(\frac{\bar{X} - 0}{\frac{c - 0}{\sqrt{n}}} \mid \mu = 0\right) = \Pr(Z > \sqrt{n}c)
\]

From the Z-table, I find that we must have \(\sqrt{n}c = 1.65\), so \(c = \frac{1.65}{\sqrt{n}}\). Thus the final form of our test is that we reject the null if \(\bar{X} > \frac{1.65}{\sqrt{n}}\).

**Problem 5**

(a) \(H_0: \mu = 7\) versus \(H_1: \mu = 6\).

(b) The test statistics is sample mean: \(T = \bar{X}\). And under the null \(\bar{X} \sim N(7, 1/\sqrt{10})\), and under the alternative \(\bar{X} \sim N(7, 1/\sqrt{10})\). We reject \(H_0\) if

\[
\bar{X} < 7 - \frac{1}{\sqrt{10}}\Phi^{-1}(0.05)
\]

\[
= 7 - \frac{1}{\sqrt{10}}(1.65)
\]

\[
= 7 - (0.32)(1.65)
\]

\[
= 6.47
\]

So we reject \(H_0\) because \(\bar{X} = 6.2 < 6.47\).

(c)

\[
\text{Power} = 1 - \beta = 1 - P(\text{don’t reject } H_0 \mid H_A \text{ is true})
\]

\[
= 1 - \left[1 - \Phi\left(\frac{6.47 - 6}{1/\sqrt{10}}\right)\right]
\]

\[
= \Phi\left(\frac{6.47 - 6}{1/\sqrt{10}}\right)
\]

\[
= \Phi(1.49)
\]

\[
= 0.93
\]

(d) With unknown \(\sigma^2\), the test statistic under the null is

\[
T = \frac{\bar{X} - \mu}{S/\sqrt{n}} \sim t_{(10-1)}
\]
we reject $H_0$ if

$$\frac{\bar{X} - \mu}{S/\sqrt{n}} < t_\alpha,$$ 

where $\alpha$ is a significance level

$$\Leftrightarrow$$

$$\frac{6.2 - 7}{\sqrt{1.5/10}} = -2.5 < -1.83 \text{ (from the table)}$$

So we reject $H_0$.

(e) In this case, the test then becomes two-sided $[H_0 : \mu = 7 \text{ versus } H_1 : \mu \neq 7]$, so all the procedures used before should be modified accordingly.

**Problem 6**

Suppose $X_i \sim N(\mu_X, 1), i = 1, \ldots, n_X$ and $Z_j \sim N(\mu_Z, 1), j = 1, \ldots, n_Z$ and all the observations are independent. You want to test the hypothesis that the means are equal against the alternative that they are not. Use the statistic

$$T = \frac{(\bar{X} - \bar{Z})}{\sqrt{1/n_X + 1/n_Z}}.$$ 

(a) The test statistic $T$ can take any real number, and if the absolute value of $T$ is ‘sufficiently’ different from 0, we will reject the null hypothesis that the means are equal.

(b) Note that $T$ is a linear transformation of $(n_X + n_Z)$ normal random variables - hence, we can guess that $T \sim N(E[T], \Var[T]).$ The mean and variance of $T$ are:

$$E[T] = E\left[\frac{(\bar{X} - \bar{Z})}{\sqrt{1/n_X + 1/n_Z}}\right]$$

$$= \frac{1}{\sqrt{1/n_X + 1/n_Z}} E[(\bar{X} - \bar{Z})]$$

$$= \frac{1}{\sqrt{1/n_X + 1/n_Z}} (E[\bar{X}] - E[\bar{Z}])$$

$$= \frac{1}{\sqrt{1/n_X + 1/n_Z}} \left( E\left[\frac{1}{n_X} \sum X_i\right] - E\left[\frac{1}{n_Z} \sum Z_j\right] \right)$$

$$= \frac{1}{\sqrt{1/n_X + 1/n_Z}} (\mu_X - \mu_Z)$$

$$= 0 \text{ (under the null } \mu_X = \mu_Z)$$
\[
Var[T] = Var \left[ \frac{\bar{X} - \bar{Z}}{\sqrt{1/n_X + 1/n_Z}} \right] \\
= \left( \frac{1}{1/n_X + 1/n_Z} \right) Var[\bar{X} - \bar{Z}] \\
= \left( \frac{1}{1/n_X + 1/n_Z} \right) (Var[\bar{X}] + Var[\bar{Z}]) \quad \text{(independence)} \\
= \left( \frac{1}{1/n_X + 1/n_Z} \right) \left( \frac{1}{n_X} \sum X_i + \frac{1}{n_Z} \sum Z_j \right) \\
= \left( \frac{1}{1/n_X + 1/n_Z} \right) \left( \frac{1}{n_X} n_X(1) + \frac{1}{n_Z} n_Z(1) \right) \\
= \left( \frac{n_X + n_Z}{n_X n_Z} \right) \left( \frac{1}{n_X} + \frac{1}{n_Z} \right) \\
= \left( \frac{n_X n_Z}{n_X + n_Z} \right) \left( \frac{n_X + n_Z}{n_X n_Z} \right) \\
= 1
\]

So \( T \sim N(0, 1) \).

(c) Since the distribution of the test statistic \( T \) is the standard normal under the null, we reject the null when \(|t| > z_{\alpha/2}\).

**Problem 7**

(a) Consider the test statistic \( T = \frac{S_X^2}{S_Y^2} \), where \( S_X^2 = \frac{1}{n_X - 1} \sum_{i=1}^{n_X} (X_i - \bar{X})^2 \) and \( S_Y^2 \) is defined analogously. We know that \( \frac{(n_X - 1)S_X^2}{\sigma_X^2} \sim \chi^2_{n_X - 1} \) and \( \frac{(n_Y - 1)S_Y^2}{\sigma_Y^2} \sim \chi^2_{n_Y - 1} \), and that these two statistics are independent. Under the null hypothesis, \( \sigma_X^2 = \sigma_Y^2 \), so we can rewrite \( T \) as \( \frac{\frac{(n_X - 1)S_X^2}{\sigma_X^2}}{\frac{(n_Y - 1)S_Y^2}{\sigma_Y^2}} \sim F(n_X - 1, n_Y - 1) \). So we reject for \( T > c \), where \( c \) is defined by \( \Pr (F(n_X - 1, n_Y - 1) > c) = \alpha = 0.10 \).

(b) We have \( n_X = 6, n_Y = 4, \bar{X} = 12, \bar{Y} = 2.75 \), \( \sum_{i=1}^{n_X} (X_i - \bar{X})^2 = 118 \), and \( \sum_{i=1}^{n_Y} (Y_i - \bar{Y})^2 = 8.75 \). So \( T = \frac{118/5}{8.75/3} = 8.09 \). From the \( F \)-table, our critical value, \( c \), is equal to 5.31, so we reject the null hypothesis.