Question 1. Relaxing assumptions on WLLN (Theorem 5.5.2).

a) Show that \( p \lim X_n = \mu \) for the same assumptions on \( X_i \) as in Theorem 5.5.2 except that now we replace the independence assumption with the assumption that the \( X_i \) are uncorrelated.

b) Further relax the assumptions so that rather than all of the \( X_i \) having the same variance, assume that all of the variances are bounded by \( C < \infty \).

Question 2. Consider the density \( f_y(y) = \frac{2y}{\theta^2} \) where \( 0 < y < \theta \). \( \theta > 0 \). For parts (b) onwards assume we observe a random sample of observations \( Y_1, \ldots, Y_n \) from this distribution.

(a) show that this is indeed a density.

(b) compute the method of moments estimator for \( \theta \) based on the first moment only.

(c) compute the variance of this estimator.

(d) compute the Fisher information for estimating \( \theta \) in the sample.

(e) contrast your results in (c) and (d) and comment.

Question 3. Suppose we have a random sample from the Poisson (\( \lambda \)) distribution. The probability density function for any \( X_i \) is

\[
 f_X(x) = \frac{e^{-\lambda} \lambda^x}{x!}
\]

for \( 0 \leq \lambda < \infty \) and \( x = 1, 2, 3, \ldots \). Both the mean and variance are equal to \( \lambda \).

(a) Consider testing between \( H_0 : \lambda = \lambda_0 \) and \( H_a : \lambda = \lambda_1 > \lambda_0 \). Show that this test rejects for \( \sum X_i \) large.

(b) Compute the sampling distribution for the test statistic (required for computation of the critical value). (note: you do not have to compute the critical value explicitly).

(c) Does a UMP test of the hypothesis \( H_0 : \lambda = \lambda_0 \) and \( H_a : \lambda > \lambda_0 \) exist? Explain.

Question 4. Consider a random variable \( X \) and the transformation \( Y = a + bX \). What is the correlation between these two random variables?