1. Let $X$ and $Y$ be random variables with finite variances. Show that
\[
\min_{g(\cdot)} E(Y - g(X))^2 = E(Y - E(Y \mid X))^2,
\]
where $g(\cdot)$ ranges over all functions.

2. Let $X$ have Poisson distribution, with parameter $\theta$
\[
X \sim \text{Poisson}(\theta) \iff P\{X = j\} = \frac{e^{-\theta} \theta^j}{j!} \quad j = 0, 1, ...
\]
Let $Y \sim \text{Poisson}(\lambda)$ be independent on $X$.

(a) Show that $X + Y \sim \text{Poisson}(\lambda + \theta)$.

(b) Show that $X \mid X + Y$ is binomial with success probability \( \frac{\theta}{\theta + X} \).

*Note:* Variable $\xi$ has binomial distribution with success probability $p$ and parameter $n$ if
\[
P\{\xi = j\} = \frac{n!}{j!(n-j)!} p^j (1-p)^{n-j}; \quad j = 0, 1, \ldots, n
\]

3. Show that if a sequence of random variables $\xi_i$ converges in distribution to a constant $c$, then $\xi_i \xrightarrow{p} c$.

4. Let $\{X_i\}$ be independent Bernoulli ($p$). Then $EX_i = p$, $Var(X_i) = p(1-p)$.

Let $Y_n = \frac{1}{n} \sum_{i=1}^{n} X_i$.

(a) Show that $\sqrt{n}(Y_n - p) \Rightarrow N(0, p(1-p))$.

(b) Show that for $p \neq \frac{1}{2}$ the estimated variance $Y_n(1 - Y_n)$ has the following limit behavior
\[
\sqrt{n}(Y_n(1 - Y_n) - p(1-p)) \Rightarrow N(0, (1 - 2p)^2 p(1-p)).
\]
(c) Show that for \( p = \frac{1}{2} \)

\[
\begin{align*}
    n \left[ Y_n(1 - Y_n) - \frac{1}{4} \right] &\Rightarrow -\frac{1}{4} \chi_1^2
\end{align*}
\]

*Note:* \( \chi_1^2 \) is a chi-square distribution with 1 degree of freedom. Let \( \xi_1, \ldots, \xi_p \) be i.i.d. \( N(0, 1) \), then \( \chi_p^2 = \sum_{i=1}^{p} \xi_i^2 \).

*Curious fact:* Note that \( Y_n(1 - Y_n) \leq 1 \), that is, we always underestimate the variance for \( p = \frac{1}{2} \).

(d) Prove that if (i) \( \frac{\sqrt{n}}{\sigma} (\xi_n - \mu) \Rightarrow N(0, 1) \) (ii) \( g \) is twice continuously differentiable: \( g'(\mu) = 0, g''(\mu) \neq 0 \), then

\[
n(g(\xi_n) - g(\mu)) \Rightarrow \sigma^2 \frac{g''(\mu)}{2} \chi_1^2.
\]

*Note:* You may assume that \( g \) has more derivatives, if it simplifies your life.

5. Let \( X_1, X_2, \ldots, X_n \sim i.i.d. \ N(\mu, \sigma^2) \). Let us define

\[
\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i, \quad s_X^2 = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2;
\]

and

\[
Y_j = \frac{X_j - \mu}{\sigma}; \quad \bar{Y}_k = \frac{1}{k} \sum_{i=1}^{k} Y_i, \quad s_k^2 = \frac{1}{k-1} \sum_{i=1}^{k} (Y_i - \bar{Y}_k)^2.
\]

(a) Show that \( \frac{(n-1)s_X^2}{\sigma^2} = (n-1)s_n^2 \).

(b) Check that

\[
(k-1)s_k^2 = (k-2)s_{k-1}^2 + \frac{k-1}{k} (Y_k - \bar{Y}_{k-1})^2.
\]

(c) Prove that if \( (k-2)s_{k-1}^2 \sim \chi_{k-2}^2 \) then \( (k-1)s_k^2 \sim \chi_{k-1}^2 \).

(d) Check that \( s_2^2 \sim \chi_1^2 \).

(e) Conclude that \( \frac{n-1}{\sigma^2}s_X^2 \sim \chi_{n-1}^2 \).

*Hint:* You can use the fact proved on the lecture that random variables \( s_k^2 \) and \( \bar{Y}_k \) are independent.