14.381: Statistics

Problem Set 2

1. Let $X_1, X_2, \ldots, X_n$ be iid observations. Find minimal sufficient statistics

   (a) $f(x \mid \theta) = \frac{2x}{\theta^2}, \ 0 < x < \theta, \ \theta > 0;$
   
   (b) $f(x \mid \theta) = e^{-(x-\theta)} \cdot \exp \left\{ -e^{-(x-\theta)} \right\}, \ -\infty < x < \infty, \ -\infty < \theta < \infty;$
   
   (c) $f(x \mid \theta) = \frac{2^i}{x!(2-x)!} \theta^x (1 - \theta)^{2-x}, \ x \in \{0, 1, 2\}, \ 0 \leq \theta \leq 1.$

2. Let $X_1, \ldots, X_n$ be independent random variables with pdfs

   $$f_{X_i}(x \mid \theta) = \frac{1}{2i\theta} \text{ for } -i(\theta - 1) < x < i(\theta + 1)$$

   and zero otherwise. Find a two-dimensional sufficient statistic for $\theta$.

3. Suppose that a random variable $X$ has a Poisson distribution with unknown parameter $\lambda$. Assume that we want to estimate $\theta = e^{-2\lambda}$. Show that the only unbiased estimator of $\theta$ is $\delta(X) = 1$ if $X$ is an even integer, and $\delta(X) = -1$ if $X$ is an odd integer. Note: It is another example of an inappropriate unbiased estimator.

4. Assume $X_1, \ldots, X_n$ are iid with mean $\mu$ and variance $\sigma^2$ (both unknown). Let us estimate mean by

   $$\hat{\mu} = \sum_{i=1}^{n} \omega_i X_i$$

   (i) Under what condition is $\hat{\mu}$ unbiased?

   (ii) Among all unbiased $\hat{\mu}$ find the one with the smallest variance.

   (iii) What $\{\omega_i\}$ should lead to the smallest MSE?
5. Let $X_1, \ldots, X_n$ be iid with pdf

$$f(x \mid \theta) = \theta x^{\theta - 1} \quad 0 \leq x \leq 1, \quad \theta > 0.$$ 

(a) Find the MLE of $\theta$, and show its consistency.

(b) Find the method of moments estimator of $\theta$.

(c) Find limit distributions of for both estimators.

- *Hint 1:* Find the distribution of $Y_i = -\log X_i$.

- *Hint 2:* Use delta-method.