Filtering

In lecture 4, we introduced filtering. Here we’ll spend a bit more time deriving some common filters and showing how to use them. Recall that an ideal band-pass filter has

\[ B^*(e^{i\omega}) = \begin{cases} 1 & \lambda \in [\pi_l, \pi_h] \\ 0 & \text{otherwise} \end{cases} \]

and can be written as

\[ B^*(e^{i\omega}) = \sum_{-\infty}^{\infty} \beta^*_j e^{-i\omega j} \]

where

\[ \beta^*_j = \frac{1}{2\pi} \int_{|\omega| \in [\pi_l, \pi_h]} e^{i\omega j} d\omega \]

\[ = \frac{1}{2\pi} \int_{\pi_l, \pi_h} e^{i\omega j} d\omega + \int_{-\pi_h, -\pi_l} e^{i\omega j} d\omega \]

\[ = \frac{1}{2\pi ij} \left( e^{i\pi h j} - e^{-i\pi_l j} + e^{-i\pi h j} - e^{-i\pi_l j} \right) \]

\[ = \begin{cases} \sin(j\pi h) - \sin(j\pi l) & j \neq 0 \\ \frac{\pi_h - \pi_l}{\pi} & j = 0 \end{cases} \]

**Baxter-King**

Baxter and King (1999) proposed approximating the ideal filter with one of order \( J \) by solving

\[ \min_{B(\cdot)} \frac{1}{2\pi} \int_{-\pi}^{\pi} |B(e^{i\omega}) - B^*(e^{i\omega})|^2 d\omega \]

\[ \text{s.t.} \quad B(1) = \phi \]
where the constraint may or may not be present. We might want to impose $B(1) = 0$ so that the filtered series is stationary, or if we’re constructing a low-pass filter, we might want $B(1) = B(e^{i\theta}) = 1$ to preserve the lowest frequency movements.

The Lagrangian is

$$L = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( B^*(e^{i\omega} - \sum_{|j| \leq J} b_j e^{-i\omega j}) \right) \left( B^*(e^{i\omega} - \sum_{|j| \leq J} b_j e^{-i\omega j}) \right)' d\omega + \lambda \left( \sum_{|j| \leq J} b_j - \phi \right)$$

The first order conditions are

$$[b_k] : 0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left( B^*(e^{i\omega} - \sum_{|j| \leq J} b_j e^{-i\omega j}) \right) e^{i\omega k} d\omega + \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-i\omega k} \left( B^*(e^{i\omega} - \sum_{|j| \leq J} b_j e^{-i\omega j}) \right)' d\omega + \lambda$$

$$[\lambda] : 0 = \sum_{|j| \leq J} \beta_j - \phi$$

Using the fact that $\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i\omega(j-k)} = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$ and $\frac{1}{2\pi} \int_{-\pi}^{\pi} B^*(\omega) e^{i\omega j} = \beta_j^*$, the first order conditions for $b_j$ are

$$b_j = \beta_j^* + \frac{\lambda}{2}$$

Using the constraint to solve for $\lambda$ gives:

$$\phi = \sum_{|j| \leq J} \beta_j^* + \frac{2J + 1}{2} \lambda$$

$$\lambda = \frac{2}{2J + 1} \left( \phi - \sum_{|j| \leq J} \beta_j^* \right)$$

To summarize: the Baxter King filter of order $J$ on $[\pi_l, \pi_h]$ constrained to have $B(0) = \phi$ is given by

$$b_j = \beta_j^* + \theta$$

where

$$\beta_j^* = \begin{cases} \sin(j\pi_h) - \sin(j\pi_l) & j \neq 0 \\ \frac{\pi_j}{\pi_h - \pi_l} & j = 0 \end{cases}$$

$$\theta = \frac{1}{2J + 1} \left( \phi - \sum_{|j| \leq J} \beta_j^* \right)$$
Christiano-Fitzgerald

Christiano and Fitzgerald (1999) propose a generalization of the Baxter-King filter. The advocate choosing a finite approximation to the ideal filter by solving

$$\min_{B_t()} E[(B_t(L)x_t - B^*(L)x_t)^2]$$

where \(x_t\) is some chosen process. \(B_t(L)\) is allowed to use all data available in your sample of length \(T\). Note that \(B_t(L)\) will generally not be symmetric and will change with \(t\). As shown in problem set 1, this is equivalent to solving,

$$\min_{B_t()} \frac{1}{2\pi} \int_{-\pi}^{\pi} |B_t(e^{i\omega}) - B^*(e^{i\omega})|^2 S_x(\omega) d\omega$$

where \(S_x(\omega)\) is the spectrum of \(x_t\). The Baxter-King filter can be considered a special case of this approach where \(x_t\) is white noise, and we restrict \(B_t()\) to be time-invariant and only have \(b_j \neq 0\) for \(|j| \leq J\). Christiano and Fitzgerald argue that having \(x_t\) a random walk works well for macro time-series.

Hodrick-Prescott

Recall that the Hodrick-Prescott filter solves:

$$\min_{\tau_t} \sum (\tau_t - x_t)^2 + \lambda (\tau_{t+1} - 2\tau_t + \tau_{t-1})^2$$

For \(1 < t < T - 1\), the first order conditions are:

$$0 = 2(\tau_t - x_t) + 2\lambda(-2(\tau_{t+1} - 2\tau_t + \tau_{t-1}) + \tau_t - 2\tau_{t-1} + \tau_{t-2} + \tau_{t+2} - 2\tau_{t+1} + \tau_t)$$

$$0 = -x_t + \lambda \tau_{t-2} - 4\lambda \tau_{t-1} + (6\lambda + 1)\tau_t - 4\lambda \tau_{t+1} + \lambda \tau_{t+2}$$

The first order conditions for \(t = 0, 1, T - 1, T\) are similar. Writing them all in matrix form, we have

$$\tau = \begin{bmatrix} \lambda & -2\lambda + 1 & \lambda & 0 & 0 & \ldots & 0 \\ -2\lambda & 5\lambda + 1 & -4\lambda & \lambda & 0 & \ldots & 0 \\ \lambda & -4\lambda & 6\lambda + 1 & -4\lambda & \lambda & \ldots & 0 \\ \vdots & & & \ddots & & & \end{bmatrix}^{-1} X$$

We can then form our cycle as \(c_t = x_t - \tau_t\).