1 Continuity of the policy function

We are in the case of bounded returns and we are assuming strict concavity of \( F \) and convexity of \( \Gamma \).

We proved that \( V \) is strictly concave. This has one important implication about the policy:

The policy correspondence \( G(x) \) is single valued, i.e., it is a function. We will use \( g(x) \) to denote it.

To prove that \( G(x) \) is single-valued we use (1) the thm of the maximum, which shows that \( G \) is u.h.c. and (2) the fact that for single-valued correspondences u.h.c. → continuity.

1. It is useful to recall steps from thm of the maximum. To apply that theorem we only use the facts that (a) \( F(x, y) + \beta V(y) \) is continuous (by Thm 4.6) and (b) \( \Gamma(x) \) is continuous (by assumption).

Then if \( x_n \to x \) and \( y_n \in G(x_n) \) we want to find a convergent subsequence with \( y_{n_k} \to y \in G(x) \). Since \( X \) is compact, a convergent subsequence \( \{y_{n_k}\} \) must exist. Take any \( y' \in \Gamma(x) \). Then since \( \Gamma \) is l.h.c. there must be a sequence \( \{y'_{n_k}\}_{k=K}^\infty \to y \) with \( y_{n_k} \in \Gamma(x_{n_k}) \). But then

\[
F(x_{n_k}, y_{n_k}) + \beta V(y_{n_k}) \geq F(x_{n_k}, y'_{n_k}) + \beta V(y'_{n_k}) \quad \text{for all } k \geq K
\]

and taking lims we have

\[
F(x, y) + \beta V(y) \geq F(x, y') + \beta V(y')
\]

since this is true for all \( y' \) we have \( y \in G(x) \).

2. Next we want to show that u.h.c. and single valued yield continuity. Suppose \( x_n \to x \) and \( \|y_{n_k} - y\| > \delta \) for all \( k \) for some \( \delta > 0 \) for some subsequence \( \{y_{n_k}\} \). But then take the sequence \( \{x_{n_k}\} \). By (1) there must be a subsequence of \( \{y_{n_k}\} \) that converges to some \( y' \in G(x) \). Since \( G \) is single-valued we must have \( y' = y \) and since \( \|y_{n_k} - y\| > \delta \) for all \( k \) we have a contradiction.
2 Differentiability of the value function

To characterize the optimum sometimes it is useful to look at the first order condition of the problem in FE:

\[ F_y (x, y) + \beta V' (y) = 0. \]

Clearly to do so we need \( F \) to be differentiable (in its second argument), but we also need \( V \) to be differentiable. What do we know about the differentiability of \( V \)?

**Example**
Simple finite horizon problem (really 1 period!):

\[ V (x) = \max_{y \in [0,1]} y^2 - xy \]

where the initial state is \( x \geq 0 \).

If \( x \leq 1 \) we have \( V (x) = 1 - x \) if \( x > 1 \) \( V (x) = 0 \).

Value function is not differentiable at \( x = 1 \), why? Lack of concavity → discontinuity in the policy function.

So we hope that concavity can give us continuity of the policy can also give us differentiability.

**Fact 1.** If a function \( f : X \to R \) is concave (with \( X \) convex subset of \( R^n \)) the function \( f \) admits a subgradient \( p \in R^n \), i.e. a \( p \) such that

\[ f(x) - f(x_0) \leq p \cdot (x - x_0) \text{ for all } x \in X. \]

Notice:

- This fact is true whether or not \( f \) is differentiable.
- If \( f \) is differentiable then \( p \) is unique and is the gradient of \( f \) at \( x_0 \).

The converse of the second point is also true:

**Fact 2.** If \( f \) is concave and has a unique subgradient, then \( f \) is differentiable.

We can now prove differentiability of \( V \)

**Theorem 1** Suppose \( F \) is differentiable in \( x \) and the value function is concave, \( x_0 \in \text{int} X \) and \( y_0 = g(x_0) \in \text{int} \Gamma (x_0) \), then \( V \) is differentiable at \( x_0 \) with

\[ \nabla V (x_0) = \nabla F_x (x_0, y_0) \]

**Proof.** The idea is to find a concave function \( W (x) \) that is a lower approximation for \( V (x) \) in a neighborhood of \( x_0 \) and that is differentiable. Let us use

\[ W (x) = F (x, y_0) + \beta V (y_0) \]

Given continuity of \( \Gamma \) and the fact that \( y_0 \in \text{int} \Gamma (x_0) \), we can find a neighborhood \( D \) of \( x_0 \) such that \( y_0 \in \Gamma (x) \) for all \( x \in D \). Then

\[ W (x) \leq V (x) \text{ for all } x \in D \]
and

\[ W(x_0) \leq V(x_0). \]

Since \( V \) is concave it has a subgradient \( p \) and we have the chain of inequalities

\[ W(x) - W(x_0) \leq V(x) - V(x_0) \leq p \cdot (x - x_0) \text{ for all } x \in D. \]

Since \( W \) is differentiable in \( x \) it must be

\[ p = \nabla W(x_0) \]

so the subgradient is unique, and, by fact 2, \( V \) is differentiable. \( \blacksquare \)