1 Asset Prices: overview

- Euler equation
- C-CAPM
- equity premium puzzle and risk free rate puzzles
- Law of One Price / No Arbitrage
- Hansen-Jagannathan bounds
- resolutions of equity premium puzzle

2 Euler equation

- agent problem
  \[
  \max \sum_{j=0}^{\infty} \sum_{s^j} \beta^j u\left(c_t\left(s^j\right)\right) Pr\left(s^j\right)
  \]

  \[
  c_t\left(s^j\right) + q_{t}^a\left(s^j\right) \cdot a_{t+1}\left(s^j\right) \leq W_t\left(s^j\right)
  \]

  \[
  W_{t+1}\left(s^{j+1}\right) = y_{t+1}\left(s^{j+1}\right) + \left(q_{t+1}^a\left(s^{j+1}\right) + d_{t+1}\left(s^{j+1}\right)\right) a_{t+1}\left(s^j\right)
  \]

- comment: \(a_t\) and \(q_t^a\) are vectors of length equal to the number of assets

- Euler equation
  \[
  u'\left(c_t\right) q_{t}^a = \beta E_t\left[u'\left(c_{t+1}\right) \left(q_{t+1}^a + d_{t+1}\right)\right]
  \]

  \[
  u'\left(c_t\right) = \beta E_t\left[u'\left(c_{t+1}\right) R_{t+1}^i\right]
  \]

  \[
  1 = E_t\left[\frac{\beta u'\left(c_{t+1}\right)}{u'\left(c_t\right)} R_{t+1}^i\right]
  \]

- transversality condition
  \[
  \lim_{j \to \infty} \beta^j E_0\left[u'\left(c_{t+j}\right) q_{t+j}^a a_{t+j}\right] = 0
  \]

- pricing formula
  repeated substitution of (1)

  \[
  q_{t}^a = \sum_{j=1}^{\infty} \beta^j E_t\left[\frac{u'\left(c_{t+j}\right) d_{t+j}}{u'\left(c_t\right)}\right]
  \]

- no bubbles
transversality and \( s_t = 1 \)

- complete markets consistency check
  review A-D price with complete markets

\[
q_{t+j}^i(s^i, s^j) = \beta \frac{u'(c^{i+1}_{t+j}(s^i, s^j))}{w'(c^i_t(s^i))} \Pr(s^j|s^i)
\]

\( \rightarrow (3) \)

3 CCAPM (Consumption Capital Asset Pricing Model)

- make (2) and (3) operational:

  CCAPM \( \equiv \) use aggregate consumption in above equations

- justifications:
  - equilibrium of representative agent economy (Lucas / Breeden)
  - equilibrium with complete markets (Constantinides)

  complete markets \( \iff \) Pareto Optima \( \iff \) representative consumer (weighted utility)

- back to Euler equation

\[
1 = E_t \left[ \beta \frac{u'(c_{t+1})}{u'(c_t)} R^i_{t+1} \right]
\]

- Absence of arbitrage implies that there exists some \( m_{t+1} \) such that

\[
1 = E_t \left[ m_{t+1} R^i_{t+1} \right]
\]

THE empirically testable condition (again)

- intuitive decomposition

\[
1 = \beta E_t \left( \frac{u'(c_{t+1})}{u'(c_t)} \right) E_t \left( R^i_{t+1} \right) + \beta \text{cov}_t \left( \frac{u'(c_{t+1})}{u'(c_t)}, R^i_{t+1} \right)
\]

\( \rightarrow \) its the covariance that matters!

4 Equity Premium Puzzle

- Euler equations with data on \( R^\text{stock market} \) and \( R^\text{bonds} \)
• simple log-normal calculation

• preferences and consumption

\[ u'(c) = c^{-\gamma} \]

\[ \frac{c_{t+1}}{c_t} = \bar{c}_\Delta \exp \left\{ \varepsilon_c - \frac{1}{2} \sigma_c^2 \right\} \]

\[ \varepsilon_c \sim N(\mu_c, \sigma_c^2) \]

\[ \Rightarrow E \left( \frac{c_{t+1}}{c_t} \right) = \mu^c \]

• returns

\[ R^i = (1 + \bar{r}^i) \exp \left\{ \varepsilon_i - \frac{1}{2} \sigma_i^2 \right\} \]

\[ \varepsilon_i \sim N(\mu_i, \sigma_i^2) \]

\[ \Rightarrow E(R^i) = R_i = 1 + \bar{r}^i \]

• Euler

\[ 1 = \beta E \left[ R^i \left( \frac{c_{t+1}}{c_t} \right)^{-\gamma} \right] \]

\[ 1 = \beta (1 + \bar{r}^i) (\bar{c}_\Delta)^{-\gamma} E \exp \left( \varepsilon_i - \frac{1}{2} \sigma_i^2 - \gamma \varepsilon_c + \gamma^2 \frac{1}{2} \sigma_c^2 \right) \]

\[ 1 = \beta (1 + \bar{r}^i) (\bar{c}_\Delta)^{-\gamma} E \exp \left( (1 + \gamma) \gamma \frac{1}{2} \sigma_c^2 - \gamma \sigma_{ie} \right) \]

• taking logs...

\[ \log (1 + \bar{r}^i) = -\log \beta + \gamma \log \bar{c}_\Delta - (1 + \gamma) \gamma \frac{1}{2} \sigma_c^2 + \gamma \sigma_{ie} \]

• stocks and bonds:

\[ \bar{r}^f \approx \log (1 + \bar{r}^f) = -\log \beta + \gamma \log \bar{c}_\Delta - (1 + \gamma) \gamma \frac{1}{2} \sigma_c^2 \] (4)

\[ \bar{r}^s \approx \log (1 + \bar{r}^s) = -\log \beta + \gamma \log \bar{c}_\Delta - (1 + \gamma) \gamma \frac{1}{2} \sigma_c^2 + \gamma \sigma_{sc} \] (5)

• premium:

\[ \bar{r}^s - \bar{r}^f \approx \log (1 + \bar{r}^s) - \log (1 + \bar{r}^f) = \gamma \sigma_{sc} \] (6)
Table removed due to copyright restrictions.


- US data (from Mehra and Prescott):

  \[ \bar{r}^s = 7\% \]
  \[ \bar{r}^f = 1\% \]
  \[ \sigma_{rc} = 0.219\% \]

- Kocherlakota
- need \( \gamma = 27 \) to match (6)
  equity premium puzzle
- to match (4) we need \( \gamma \) very high or very low risk free rate puzzle
5 Discount Factors: LOP and NA

I follow Cochrane and Hansen (1992) closely – great paper to read

- two periods "now" and "then" (t and t + 1 if you prefer)
- J “fundamental” assets:
  - $x_j$ payoff “then”
  - $q_i$ “now” price

$→$ stack into $x$ and $q$ (column) vectors

- payoff space for "then" $P ≡ \{ p : p = c \cdot x \text{ for some } c \in \mathbb{R} \}$

- pricing function $\pi (p) : P \to \mathbb{R}$
  then $\pi (x) = q$
• definition: Law of One Price (LOP) holds if the pricing function is linear

\[ \pi (c \cdot x) = c \cdot \pi (x) = c \cdot q \]

\[ \Rightarrow c \cdot x = c' \cdot x \text{ then } c \cdot q = c' \cdot q \]

• definition: discount factor \( y \in P \)

\[ \pi (p) = E (yp) \]

• **Riesz representation Theorem**

LOP \( \iff \exists \) (stochastic) discount factor \( y \in P \)

• Let \( Y \) be the set of all discount factors

• note: \( y \) may be negative

• example:

\[ y^* = x' (Exx')^{-1} q \]

note: if \( Exx' \) is non-singular then remove assets from \( x \) until it is!

a non-singular \( Exx' \) means that (a) there is a risk-free asset (b) there are two ways of getting the same payoff

• Definition: No Arbitrage (NA) holds

\[ p \geq 0 \Rightarrow \pi (p) \geq 0 \]

\[ p > 0 \text{ (with positive prob.)} \Rightarrow \pi (p) > 0 \]

• result NA \( \iff \exists \) strictly positive discount factor \( y > 0 \)

Let \( Y^{++} \) be the set of all discount factors that are positive

• examples

\[ m = \frac{\beta^t u' (c_{then})}{u' (c_{now})} \]

\[ \text{proof:} \]

\[ \pi (c \cdot x) = \pi (c' \cdot x) \]

\[ c\pi (x) = c' \pi (x) \]

\[ cq = c' q \]
6 Hansen-Jagannathan bounds

• all theories:

\[
q = E(mp) \\
m = f(\text{data, parameters})
\]

(see Cochrane’s book)

• note \(p^i/q^i\) is rate of return

• H-J bounds:

   diagnostic tool for models of \(m\)

• special case:

   data on a single excess return relative to some baseline asset

\[
r = p/q - p^0/q^0
\]

then \(\pi (r) = 0\) so that

\[
0 = Emr = EmEr + cov(m, r) \\
= EmEr + \sigma_m \sigma_r corr(m, r)
\]

\[-1 \leq \frac{EmEr}{\sigma_m \sigma_r} = corr(m, r) \leq 1\]

\[
\left| \frac{EmEr}{\sigma_m \sigma_r} \right| \leq 1 \\
\frac{\sigma_r}{|Er|} \leq \frac{\sigma_m}{Em}
\]

intuition: need volatile \(\sigma_m\)

• note: \(Em = 1/R^f\) if there is a risk free rate \(R^f\)

• lets generalize:

   for any random vector \(x\) we can consider the population regression:

   \[
m = a + x'b + e
\]

   which just defines \(e\) uniquely as having \(\mathbb{E}e = 0\) and \(cov(x, e) = 0\)

• by definition \(cov(e, x) = 0\)

   \[
   \Rightarrow \text{var} (m) \geq \text{var} (x'b)
   \]
• idea compute $x'b$ and $\text{var}(x'b)$ to get lower bound
  → check whether theories for $y$ pass this test

  $$b = [\text{cov}(x, x)]^{-1} \text{cov}(x, y)$$
  $$a = Ey - Ex'b$$

• How to compute $b$?
  idea: if $x = p$ then theory helps...

• assume $x = p$ note that
  $$\text{cov}(x, y) = q - E(y)E(x)$$
  so:
  $$b = [\text{cov}(x, x)]^{-1} [q - E(y)E(x)]$$
  $$\text{var}(x'[\text{cov}(x, x)]^{-1} [q - E(y)E(x)]) = \text{var}(x)[\text{var}(x)]^{-2} E(y)^2 E(x)^2$$

• if we knew $E(y)$ we have a lower bound
  otherwise $\Rightarrow$ feasible region for pair $(E(y), \text{var}(y))$

• convenient
  - no need to recompute lower bound for each theory
  - helps see where the theory fails

• 3 cases:
  - risk-less return
    → $E(y)$ pinned down and risky return
  - one excess-return $q = 0$
    Sharpe ratio and market price of risk (what we did before!)
  - general case $\Rightarrow$ very flexible, see CH paper

• figures 2.1: excess

• 2.2, 2.3, 2.4 from CH paper
7 Resolutions (?)

7.1 Exotic Preferences

- Risk Aversion vs. IES
  (Weil / Epstein-Zin)
- first-order risk aversion
  (Epstein-Zin)
- habit persistence \( e.g. u(c_t - \alpha c_{t-1}) \)
  (Abel / Campbell-Cochrane)
- loss-aversion

7.2 Heterogenous Agent Incomplete Markets

- uninsured idiosyncratic risk
  (Mankiw / Constantinides-Duffie)
- borrowing constraints (Euler with inequality)
  (Luttmer / Heaton-Lucas)
- constrained optima with limited commitment
  (Alvarez-Jermann)

7.3 Knightian Uncertainty

- risk vs. uncertainty
- fear of not understanding returns / uncertainty over probability distribution / desire for robust decisions (Hansen and Sargent)

7.4 No risk premium!

- no risk premium to explain...
- historical returns on stocks were unexpected
  (McGratten-Prescott)
- bonds are money \( \rightarrow \) low return
- stocks more risky than sample (low probability of a crash)
  (see Reitz, Cochrane, Weitzman and Barro)
8 Conclusions

Risk premium puzzle

- great example of interplay between theory and data
- no strong consensus on resolution yet
  many new ideas
- new models should explore
- revisit the welfare costs of BCs
  (Alvarez and Jermann)