When does it pay to use two technologies at the same time?
Assume that output can be produced using two technologies, labelled by 1 and 2. The production functions are given by

\[ Y_1 = K_1^\alpha L_1^{1-\alpha} \]

\[ Y_2 = K_2^\beta L_2^{1-\beta} \]

1. Show that if the two technologies are used, the capital labor ratio in technology 2 must be equal to

\[ k_2 = \frac{K_2}{L_2} = \left( \frac{\alpha}{\beta} \right)^{\frac{\beta}{\alpha}} \left( \frac{1 - \alpha}{1 - \beta} \right)^{\frac{1 - \alpha}{1 - \beta}} \]

\[ k_1 = \frac{K_1}{L_1} = \left( \frac{\beta}{\alpha} \right)^{\frac{\alpha}{\beta}} \left( \frac{1 - \beta}{1 - \alpha} \right)^{\frac{1 - \beta}{1 - \alpha}} \]

Show that this entirely determines factor prices and the allocation of labor and capital between the two technologies.

2. Show that if \( \alpha = 1/3, \beta = 2/3 \), total capital is \( K \) and total labor is \( L \), then

\[ k_2 = 2 \]
\[ k_1 = 1/2 \]
\[ L_2 = \frac{2K - L}{3} \]
\[ L_1 = \frac{4L - 2K}{3}. \]

3. Between which bounds must the aggregate \( K/L \) lie for the two technologies to be used in equilibrium?

4. Show that if \( K/L \) is between these two bounds, then they will actually be used in equilibrium.

5. Can you explain what is going on in the \((K, L)\) plane?