Problem Set 4 Solution

1. It is easy to check that \( p_t = \varepsilon_t \) is the (bubbleless) REE equilibrium. Next consider the learning process. Substituting the model equation into the learning equation yields

\[
p_{t+1}^c = p_t^c + \frac{1}{g_t} (\alpha p_t^c + \varepsilon_{t-1} - p_t^c) = h_t p_t^c + \frac{\varepsilon_{t-1}}{g_t}.
\]

By induction one obtain

\[
p_{t+1}^c = p_0^c \prod_{s=0}^{t} h_s + \sum_{s=1}^{t+1} \frac{\varepsilon_{s-2}}{g_{s-1}} \prod_{\tau=s}^{t} h_\tau
\]

Using this formula and the fact that the \( \varepsilon_t \) are iid, we can compute

\[
\mathbb{E} [p_{t+1}^c - \mathbb{E}_{t,\text{REE}} p_{t+1}^c]^2 = (p_0^c)^2 \prod_{s=0}^{t} h_s + \sigma^2 \sum_{s=1}^{t+1} \frac{1}{g_{s-1}} \prod_{\tau=s}^{t} h_\tau = (p_0^c)^2 \prod_{s=0}^{t} h_s + \sigma^2 m_t
\]

2. Necessity is obvious. To show sufficiency notice that \( m_t \geq \prod_{s=1}^{t+1} h_s^2 \), so \( \lim_{t \to \infty} m_t = 0 \) insures that the term in front of \( p_0^c \) vanishes as \( t \to \infty \).

3. We have

\[
h_{t+1}^2 m_t + \frac{1}{g_{t+1}^2} = h_{t+1}^2 \sum_{s=1}^{t+1} \frac{1}{g_{s-1}} \prod_{\tau=s}^{t} h_\tau^2 + \frac{1}{g_{t+1}^2} = \sum_{s=1}^{t+1} \frac{1}{g_{s-1}} \prod_{\tau=s}^{t} h_\tau^2 + \frac{1}{g_{t+1}^2} = \sum_{s=1}^{t+2} \frac{1}{g_{s-1}} \prod_{\tau=s}^{t+1} h_\tau^2 = m_{t+1}
\]

4. First notice that \( m_t \geq \frac{1}{g_t^2} \) so \( \lim_{t \to \infty} m_t = 0 \) clearly requires that \( \lim_{t \to \infty} g_t = +\infty \). Also \( m_t \geq \frac{1}{g_{s-1}} \prod_{\tau=s}^{t} h_\tau^2 \) for all \( s \in \{1, \ldots, t+1\} \). It follows that \( \lim_{t \to \infty} m_t = 0 \) requires

\[
\lim_{t \to \infty} \prod_{\tau=s}^{t} h_\tau = 0
\]

for all \( s \geq 1 \), which is stronger than the requirement stated in the problem set since it must hold for all \( s \geq 1 \). Finally if \( \alpha \geq 1 \), then \( h_t \geq 1 \) for all \( t \) which is inconsistent with \( \lim_{t \to \infty} \prod_{\tau=s}^{t} h_\tau = 0 \). Thus it must be the case that \( \alpha < 1 \).
5. Suppose
\[
\lim_{t \to \infty} \prod_{\tau=s}^{t} h_{\tau} = 0
\]
for all \( s \geq 1 \). The goal is to show that \( \sum_{0}^{+\infty} \frac{1}{g_t} = +\infty \).

6. Pick \( T \) such that \( t \geq T \) implies \( \frac{1-a}{q_t} < \frac{1}{2} \) and define \( a_n = \frac{1-a}{g_{T+n}} \) for \( n \geq 0 \). We want to show that \( \prod_{n=0}^{\infty} (1-a_n) = 0 \) if and only if \( \sum_{n=0}^{\infty} a_n = +\infty \). This follows from some results on convergence of infinite products, see the book *Mathematical Analysis, 2ed* by Apostol, pp. 206–209. One direction is not so difficult: as \( 1-a_n \leq e^{-a_n} \), we have \( \prod_{n=0}^{\infty} (1-a_n) \leq e^{-\sum_{n=0}^{\infty} a_n} \). At least to me it seems that the other direction is a bit more involved, but perhaps you found an easy proof. Otherwise take a look at the book by Apostol.

7. If \( g_t \) is a constant, then \( \lim_{t \to \infty} g_t = +\infty \) fails. If \( g_t = \frac{1}{(t+1)^2} \), then \( \sum_{0}^{+\infty} \frac{1}{g_t} = +\infty \) fails.

8.