Problem Set 5
(due April 12)

Problem 1
There is a measure-one continuum of agents, indexed by $i \in [0, 1]$. Each agent can choose between two actions. The action of agent $i$ is denoted as $k_i \in \{0, 1\}$, where $k_i = 0$ represents “not invest” and $k_i = 1$ represents “invest”. All agents move simultaneously. The utility of agent $i$ is given by

$$u_i = U(k_i, K, \theta) = e^{\theta k_i} (1 + K^\gamma) - c k_i$$

where $\theta$ reflects “fundamentals” and $K = \int k_i di$ denotes the mass of agents investing.

1. Suppose that $\theta$ is commonly known by all agents. What is the best response $g(K, \theta)$ (agents are assumed not invest in case of indifference)? Derive the thresholds $\theta$ and $\bar{\theta}$ such that: (i) all agents not investing is the unique equilibrium for $\theta < \theta$; (ii) all agents investing is the unique equilibrium for $\theta \geq \bar{\theta}$ and (iii) for intermediate values $\theta \in [\theta, \bar{\theta})$ there are multiple equilibria.

From now on assume that agent $i$ observes a private signal

$$x_i = \theta + \xi_i$$

with $\xi_i \sim \mathcal{N}(0, \sigma_x^2)$. All agents also observes an exogenous public signal

$$z = \theta + \varepsilon$$

where $\varepsilon \sim \mathcal{N}(0, \sigma_z^2)$. Let $\alpha_x = \sigma_x^{-2}$ and $\alpha_z = \sigma_z^{-2}$. Agents have a common prior about $\theta$, which is uniform over the entire real line. Equilibrium is defined as follows: a strategy $k(\cdot)$ and an aggregate investment $K(\cdot)$ constitute an equilibrium if

$$k(x, z) \in \arg \max_k \mathbb{E}[U(k, K(\theta, z), \theta|x, z)]$$

$$K(\theta, z) = \int k(x, z) \sqrt{\alpha_x} \phi(\sqrt{\alpha_x} [x - \theta]) dx.$$ 

The remainder of the exercise asks you to numerically compute monotone equilibria, that is, equilibria in which $k(x, z)$ is monotone in $x$. In a monotone equilibrium, for any realization of $z$, there is a threshold $x^*(z)$ such that an agent invests if and only if $x \geq x^*(z)$. Throughout, set $\gamma = 0.8$ and $c = 2$. 

2. Set $\alpha_x = 10$ and $\alpha_z = 1$. Over a range of values for $z$ (somewhat wider than $[\bar{\theta}, \theta]$) plot the equilibrium thresholds $x^*(z)$. Are there values of $z$ for which there are multiple equilibrium thresholds?

3. Repeat part 2. for the values $\alpha_x = 1$ and $\alpha_z = 10$. Compute the range $[\underline{z}, \bar{z}]$ of values of $z$ for which there is multiplicity.

4. Fix $\alpha_x$ at one. What happens to the range $[\underline{z}, \bar{z}]$ as $\alpha_z$ becomes large?

Problem 2
Consider the following two stage version of Morris-Shin. The utility of agent $i$ is given by

$$u_i = a_{1,i}(Rb - c) + \beta a_{2,i}(Rb - c)$$

Here $a_{1,i} \in \{0, 1\}$ is the action of agent $i$ in the first stage and $a_{2,i} \in \{0, 1\}$ is the action of agent $i$ in the second stage. For both actions, a value of one represents “attack” while a value of zero represents “not attack”. The regime outcome is denoted as $R$, where $R = 0$ represents survival of the status quo and $R = 1$ represents collapse. The decision to attack is irreversible: $a_{1,i} = 1$ implies $a_{2,i} = 1$. Let $A_1 = \int a_{1,i} di$ be mass of agents attacking in the first stage. Let $A_2 = \int a_{2,i}(1 - a_{1,i}) di$ be the mass of agents that did not attack in the first stage but joined the attack in the second stage. The timing is as follows. First agent $i$ observes a private signal $x_i = \theta + \xi_i$ where $\xi_i \sim N(0, \sigma_x^2)$ and $\alpha_x = \sigma_x^{-2}$. Then agents simultaneously choose their first stage actions $a_{1,i}$. Then agents observe the public signal $z = \Phi^{-1}(A_1) + v$ where $v \sim N(0, \sigma_z^2)$ and $\alpha_z = \sigma_z^{-2}$. The status quo collapses ($R = 1$) if $A_1 + A_2 \geq \theta$. The goal is to characterize the monotone equilibria of this model.

1. First note that agent $i$ will attack in the first period if $x_i \leq x_1^*$ for some threshold $x_1^*$. What is the mass of agents choosing to attack in the first period as a function of $\theta$, denoted as $A_1(\theta)$? Plug the function $A_1(\theta)$ into the expression for the public signal $z$. What are the properties of $z$ as a signal about $\theta$?

2. Given a first period threshold $x_1^*$, characterize the second stage equilibrium thresholds $x_2^*(z)$. Under what condition on $\alpha_z$ and $\alpha_x$ is their multiplicity?

3. Now move back to the first stage and analyze the monotone equilibria of the model. Discuss multiplicity and how it is related to the precision parameters $\alpha_z$ and $\alpha_x$ and the discount rate $\beta$. 